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Genus One Partition Function of the Calabi-Yau d-Fold embedded in CP^{d+1}

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ABSTRACT

For a one-parameter family of Calabi-Yau d-fold M embedded in CP^{d+1}, we consider a new quasi-topological field theory A^*(M)-model compared with the A(M)-model. The two point correlators on the sigma model moduli space (the hermitian metrics) are analyzed by the AA^*-fusion on the world sheet sphere. A set of equations of these correlators turns out to be a non-affine A-type Toda equation system for the d-fold M. This non-affine property originates in the vanishing first Chern class of M. Using the results of the AA^*-equation, we obtain a genus one partition function of the sigma model associated to the M in the recipe of the holomorphic anomaly. By taking an asymmetrical limit of the complexified Kähler parameters $\bar{t} \to \infty$ and t is fixed, the A^*(M)-model part is decoupled and we can obtain a partition function (or one point function of the operator $\mathcal{O}^{(1)}$ associated to a Kähler form of M) of the A(M)-matter coupled with the topological gravity at the stringy one loop level. The coefficients of the series expansion with respect to an indeterminate $q := e^{2\pi i t}$ are integrals of the top Chern class of the vector bundle $\mathcal{V}$ over the moduli space of stable maps with definite degrees.

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1 Introduction

When one considers correlation functions of an N=2 non-linear sigma model with a Calabi-Yau target space, they depend not only on the moduli space of the Riemann surface, but also on the properties of the target Calabi-Yau manifold (especially on the Calabi-Yau moduli spaces). By twisting the N=2 non-linear sigma model [1, 2], one can obtain two quasi-topological field theories (the A-model and the B-model) [3], which describe two distinct Calabi-Yau moduli spaces (a Kähler structure moduli space and a complex structure moduli space, respectively).

So far it was difficult to analyze non-perturbative corrections of the A-model correlation functions. However by the discovery of the mirror symmetry [4, 5] between the A(M)-model and the B(W)-model for the mirror pairs (M, W), it is becoming possible to investigate these corrections in the A-model [6, 7].

Up to now, three point functions and d-point functions on the genus 0 Riemann surface (i.e. at the stringy tree level) for the Calabi-Yau d-folds [8, 9, 10, 11, 12, 13] and partition functions in the higher genus ($g \geq 1$) (i.e. at the stringy higher (more than or equal to one) loop level) for Calabi-Yau 3-folds [14] have been investigated under mirror symmetries. Now it may fairly be said that the analyses of the Calabi-Yau 3-folds under the mirror symmetries have been established [15, 16, 17, 18, 19, 20, 21].

In this article, we take a genus one Riemann surface as a world sheet, a (complex) d-dimensional Calabi-Yau target space M realized as a hypersurface embedded in a projective space CP^{d+1} and study the properties of a partition function of the A(M)-model under the mirror symmetry.

The rest of the paper is organized as follows. In section 2, we review an $N = 2$ supersymmetric non-linear sigma model with a Calabi-Yau target space M and its topological version A(M)-model [1, 22]. Also another topological theory A^*(M)-model is introduced. In section 3, a one-parameter family of the Calabi-Yau d-fold M and its mirror pair W are explained. In order to study the hermitian metrics $\mathsf{g}_{l\bar{m}}$ of the sigma model moduli space, we develop a method AA^*-fusion there. This recipe is applied to our one-parameter family of d-fold in section 4. By using the data the author derived before, a set of equations for the metrics $\mathsf{g}_{l\bar{m}}$ is obtained. This equation system turns out to be a non-affine A-type Toda equation system. We solve this equation system explicitly and investigate the properties of the metrics. Also the results obtained there are used to analyze a genus one partition function for the d-fold. Section 5 provides a field theoretical interpretation for the genus one partition function. Section 6 is devoted to the conclusion and comments. In appendices, several explanations and calculations omitted in the text are collected.

2 Review of the A-Model

In this section, we review an A(M)-model associated with a d-dimensional Calabi-Yau manifold M. In addition, we introduce another (quasi)-topological field theory A^*-model.

2.1 $N = 2$ Supersymmetric Non-linear Sigma Model

To begin with, we introduce an $N = 2$ supersymmetric non-linear sigma model in two dimensions. Let Σ be a Riemann surface and M be a d-dimensional Calabi-Yau manifold. Locally one chooses coordinate systems $(X^i, X^{\bar{\imath}})$ on M and $(z, \bar{z})$ on Σ (where X^i stands for a set of the holomorphic coordinate system $(i = 1, 2, \cdots, d)$). The $(X^i, X^{\bar{\imath}})$ are considered as mappings from Σ to M. In this supersymmetric theory, there are fermionic pairs $(\psi_L^i, \psi_L^{\bar{\imath}})$, $(\psi_R^i, \psi_R^{\bar{\imath}})$ where subscripts "L","R" stand for left-moving parts, right-moving parts on Σ respectively. Also superscripts "i", "$\bar{\imath}$" mean that $\psi^i, \psi^{\bar{\imath}}$ have values on $X^*(T^{1,0}M)$, $X^*(T^{0,1}M)$ respectively. With these fields, the Lagrangian is written,

$$
\begin{aligned}
L_0 \;&:=\; \int_\Sigma d^2 z \Bigg[\frac{1}{2} g_{i\bar{\jmath}} \left(\partial_z X^i \partial_{\bar{z}} X^{\bar{\jmath}} + \partial_{\bar{z}} X^i \partial_z X^{\bar{\jmath}} \right) \\
&+\; \sqrt{-1} g_{i\bar{\jmath}} \psi_L^{\bar{\jmath}} D_{\bar{z}} \psi_L^i + \sqrt{-1} g_{i\bar{\jmath}} \psi_R^{\bar{\jmath}} D_z \psi_R^i + R_{i\bar{\jmath}k\bar{l}} \psi_L^i \psi_L^{\bar{\jmath}} \psi_R^k \psi_R^{\bar{l}} \Bigg] \;. \qquad (1)
\end{aligned}
$$

Here covariant derivatives $D_z, D_{\bar{z}}$ are defined as,

$$
\begin{aligned}
D_z \psi_R^I &:= \frac{\partial}{\partial z} \psi_R^I + \frac{\partial X^J}{\partial z} \Gamma^I_{JK} \psi_R^K \;\;, \\
D_{\bar{z}} \psi_L^I &:= \frac{\partial}{\partial \bar{z}} \psi_L^I + \frac{\partial X^J}{\partial \bar{z}} \Gamma^I_{JK} \psi_L^K \;\;,
\end{aligned}
$$

with Γ^I_{JK} being the Levi-Civita connection of M and the curvature $R_{i\bar{\jmath}k\bar{l}}$ is the Riemann tensor of M. This Lagrangian system possesses an $N = 2$ supersymmetry,

	Q_R	$\widetilde{Q}_R$	Q_L	$\widetilde{Q}_L$
X^i	ψ_R^i	0	ψ_L^i	0
$X^{\bar{\imath}}$	0	$\psi_R^{\bar{\imath}}$	0	$\psi_L^{\bar{\imath}}$
ψ_L^i	$-\Gamma^i_{jk}\psi_R^j\psi_L^k$	0	0	$\sqrt{-1}\partial_z X^i$
$\psi_L^{\bar{\imath}}$	0	$-\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\psi_R^{\bar{\jmath}}\psi_L^k$	$\sqrt{-1}\partial_z X^{\bar{\imath}}$	0
ψ_R^i	0	$\sqrt{-1}\partial_{\bar{z}} X^i$	$-\Gamma^i_{jk}\psi_R^j\psi_L^k$	0
$\psi_R^{\bar{\imath}}$	$\sqrt{-1}\partial_{\bar{z}} X^{\bar{\imath}}$	0	0	$-\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\psi_R^{\bar{\jmath}}\psi_L^k$

,

where Q_L, $\widetilde{Q}_L$, Q_R, $\widetilde{Q}_R$ are super charges which generate super-transformations,

$$
\delta_L \mathcal{O} = \sqrt{-1} \epsilon_L \left\{ Q_L \,, \mathcal{O} \right\} \;\;,
$$

$$
\begin{aligned}
&\tilde{\delta}_L\mathcal{O} = \sqrt{-1}\tilde{\epsilon}_L\left\{\tilde{Q}_L\,,\mathcal{O}\right\} \ , \\
&\delta_R\mathcal{O} = \sqrt{-1}\epsilon_R\left\{Q_R\,,\mathcal{O}\right\} \ , \\
&\tilde{\delta}_R\mathcal{O} = \sqrt{-1}\tilde{\epsilon}_R\left\{\tilde{Q}_R\,,\mathcal{O}\right\} \ , \\
&\quad \epsilon_L\,,\,\tilde{\epsilon}_L\,,\,\epsilon_R\,,\,\tilde{\epsilon}_R\,;\ \text{fermionic parameters} \ .
\end{aligned}
$$

These charges satisfy the following anti-commutation relations,

$$
\begin{aligned}
&Q_L^2 = \tilde{Q}_L^2 = Q_R^2 = \tilde{Q}_R^2 = 0 \ , \\
&\left\{Q_L\,,Q_R\right\} = \left\{\tilde{Q}_L\,,\tilde{Q}_R\right\} = 0 \ , \\
&\left\{\tilde{Q}_L\,,Q_R\right\} = \left\{Q_L\,,\tilde{Q}_R\right\} = 0 \ , \\
&\left\{Q_L\,,\tilde{Q}_L\right\} = \partial_z \ , \ \left\{Q_R\,,\tilde{Q}_R\right\} = \bar{\partial}_{\bar{z}} \ .
\end{aligned}
$$

2.2 Topological Sigma Model

In order to obtain topological versions of the $N = 2$ non-linear sigma model, let us consider the alternation of bundles on which fermions take values. We change spins of fermions by an amount depending on their $U(1)$ charges. As a result, fermions take values not on spin bundles but on (anti)-canonical bundles. Especially we consider two cases;

• Case I (A(M)-Model)
In this case, we change the bundles on which fermions take values and rename the resulting fields as following,

$$
\left\{
\begin{array}{lllllll}
\psi_L^i & : & K^{1/2}\otimes X^*(T^{1,0}M) & \rightarrow & \chi^i & : & K^0\otimes X^*(T^{1,0}M) \\
\psi_L^{\bar{\imath}} & : & K^{1/2}\otimes X^*(T^{0,1}M) & \rightarrow & \rho_z^{\bar{\imath}} & : & K^1\otimes X^*(T^{0,1}M) \\
\psi_R^i & : & \bar{K}^{1/2}\otimes X^*(T^{1,0}M) & \rightarrow & \rho_{\bar{z}}^i & : & \bar{K}^1\otimes X^*(T^{1,0}M) \\
\psi_R^{\bar{\imath}} & : & \bar{K}^{1/2}\otimes X^*(T^{0,1}M) & \rightarrow & \chi^{\bar{\imath}} & : & \bar{K}^0\otimes X^*(T^{0,1}M)
\end{array}
\right. ,
$$

where the K is a canonical bundle.

• Case II (A^*(M)-Model)
Similarly to the case (I), we can perform an opposite twisting,

$$
\left\{
\begin{array}{lllllll}
\psi_L^i & : & K^{1/2}\otimes X^*(T^{1,0}M) & \rightarrow & \bar{\rho}_z^i & : & K^1\otimes X^*(T^{1,0}M) \\
\psi_L^{\bar{\imath}} & : & K^{1/2}\otimes X^*(T^{0,1}M) & \rightarrow & \bar{\chi}^{\bar{\imath}} & : & K^0\otimes X^*(T^{0,1}M) \\
\psi_R^i & : & \bar{K}^{1/2}\otimes X^*(T^{1,0}M) & \rightarrow & \bar{\chi}^i & : & \bar{K}^0\otimes X^*(T^{1,0}M) \\
\psi_R^{\bar{\imath}} & : & \bar{K}^{1/2}\otimes X^*(T^{0,1}M) & \rightarrow & \bar{\rho}_{\bar{z}}^{\bar{\imath}} & : & \bar{K}^1\otimes X^*(T^{0,1}M)
\end{array}
\right. .
$$

We call the former model as A(M)-model and the latter as A^*(M)-model. As topological theories, the super charges $\tilde{Q}_L$ and Q_R are combined into a BRST charge $Q^{(+)}$,

$$
Q^{(+)} := Q_L + \tilde{Q}_R \ ,
$$

in the A(M)-model. On the other hand, a BRST charges $Q^{(-)}$ is constructed from the remaining ones Q_L and $\widetilde{Q}_R$ in the A^*(M)-model,

$$Q^{(-)} := \widetilde{Q}_L + Q_R \ .$$

Local observables of the A(M)-model, the A^*(M)-model are defined as elements of the BRST cohomologies with respect to the BRST charges $Q^{(+)}$, $Q^{(-)}$ respectively.

Firstly in the A(M)-model, local observables are functionals of the fields $(X^i, X^{\bar{\jmath}}, \chi^i, \chi^{\bar{\jmath}})$. Considering the correspondence between the de Rham cohomology and the A-model BRST cohomology, we associate an arbitrary de Rham cohomology element ω to a physical observables $\phi_A[\omega]$ in the A(M)-model,

$$\begin{aligned} & \omega = \omega_{i_1 \cdots i_p \bar{\jmath}_1 \cdots \bar{\jmath}_q} dX^{i_1} \wedge \cdots \wedge dX^{i_p} \wedge dX^{\bar{\jmath}_1} \wedge \cdots \wedge dX^{\bar{\jmath}_q} \in H^{p,q}_d(M) \ , \\ & \qquad\qquad (\, d\omega = 0 \ , \ \omega \sim \omega + d\nu \,) \ , \\ \leftrightarrow \quad & \phi_A[\omega] = \omega_{i_1 \cdots i_p \bar{\jmath}_1 \cdots \bar{\jmath}_q} \chi^{i_1} \cdots \chi^{i_p} \chi^{\bar{\jmath}_1} \cdots \chi^{\bar{\jmath}_q} \ , \\ & \qquad\qquad (\, \delta\phi_A[\omega]) = 0 \,) \ . \end{aligned}$$

Especially a relation is satisfied,

$$\{Q^{(+)}, \phi_A[\omega]\} = \phi_A[d\omega] \ .$$

Secondly A^*(M)-model observables are constructed from the fields $(X^i, X^{\bar{\jmath}}, \overline{\chi}^i, \overline{\chi}^{\bar{\jmath}})$. In similar to the A(M)-model case, we can define a physical operator $\phi_{A^*}[\omega]$ in the A^*(M)-model for each cohomology element ω,

$$\begin{aligned} & \omega = \omega_{i_1 \cdots i_p \bar{\jmath}_1 \cdots \bar{\jmath}_q} dX^{i_1} \wedge \cdots \wedge dX^{i_p} \wedge dX^{\bar{\jmath}_1} \wedge \cdots \wedge dX^{\bar{\jmath}_q} \in H^{p,q}_d(M) \ , \\ & \qquad\qquad (\, d\omega = 0 \ , \ \omega \sim \omega + d\nu \,) \ , \\ \leftrightarrow \quad & \phi_{A^*}[\omega] = \omega_{i_1 \cdots i_p \bar{\jmath}_1 \cdots \bar{\jmath}_q} \overline{\chi}^{i_1} \cdots \overline{\chi}^{i_p} \overline{\chi}^{\bar{\jmath}_1} \cdots \overline{\chi}^{\bar{\jmath}_q} \ , \\ & \qquad\qquad (\, \delta\phi_{A^*}[\omega]) = 0 \,) \ . \end{aligned}$$

Then a relation is satisfied,

$$\{Q^{(-)}, \phi_{A^*}[\omega]\} = \phi_{A^*}[d\omega] \ .$$

3 The AA^*-Fusion and the Two Point Functions

In this section, we investigate the mixing between the holomorphic part and the anti-holomorphic one (the hermitian metrics) in the Calabi-Yau non-linear sigma model.

3.1 The One-Parameter Model in CP^{d+1}

Throughout this article, we take a one-parameter family of Calabi-Yau d-fold M realized as a zero locus of a hypersurface embedded in a projective space CP^{d+1},

$$\begin{aligned} M\,;\ \mathsf{p} = X_1^{d+2} + X_2^{d+2} + \cdots + X_{d+2}^{d+2} \\ -(d+2)\psi(X_1 X_2 \cdots X_{d+2}) = 0 \ \ in \ \ CP^{d+1} \ \ , \end{aligned} \tag{2}$$

as a target space in the N=2 non-linear sigma model. Hodge numbers $h^{p,q}$ of this d-fold are calculated as [23],

$$\begin{aligned} & h^{p,q} = \delta_{p,q} \ \ , \ \ (0 \leq p \leq d \,,\, 0 \leq q \leq d \,,\, p+q \neq d) \ \ , \\ & h^{d-p,p} = \delta_{2p,d} + \sum_{i=0}^{p} (-1)^i \begin{pmatrix} d+2 \\ i \end{pmatrix} \cdot \begin{pmatrix} (p+1-i)(d+1)+p \\ d+1 \end{pmatrix} \ \ , \ \ (0 \leq p \leq d) \ \ . \end{aligned} \tag{3}$$

Especially a Euler number $\chi(M)$ can be written down,

$$\chi(M) = \frac{1}{N}\left\{ (1-N)^N - 1 + N^2 \right\} \ \ , \ \ N := d+2 \ \ .$$

A mirror manifold W paired with this M is constructed as a orbifold divided by some maximally invariant discrete group $G = (\mathbf{Z}_{d+2})^{(d+1)}$,

$$\mathrm{W}\,;\ \{\mathsf{p}=0\}/G \ \ .$$

This discrete group acts on the coordinate $(X_1 \,,\, X_2 \,,\, \cdots \,,\, X_{d+1} \,,\, X_{d+2})$ as $(\tilde{\alpha}^{a_1} X_1 \,,\, \tilde{\alpha}^{a_2} X_2 \,,\, \cdots \,,\, \tilde{\alpha}^{a_{d+1}} X_{d+1} \,,\, \tilde{\alpha}^{a_{d+2}} X_{d+2})$ with $\tilde{\alpha} := \exp\left(\frac{2\pi i}{d+2}\right)$,

$$(a_1 \,,\, a_2 \,,\, a_3 \,,\, \cdots \,,\, a_d \,,\, a_{d+1} \,,\, a_{d+2}) = \begin{cases} (1\,,0\,,0\,,\cdots\,,0\,,0\,,d+1) \ \ , \\ (0\,,1\,,0\,,\cdots\,,0\,,0\,,d+1) \ \ , \\ (0\,,0\,,1\,,\cdots\,,0\,,0\,,d+1) \ \ , \\ \qquad\qquad \cdots \\ (0\,,0\,,0\,,\cdots\,,1\,,0\,,d+1) \ \ , \\ (0\,,0\,,0\,,\cdots\,,0\,,1\,,d+1) \ \ . \end{cases}$$

When one thinks about the Hodge structure of the G-invariant parts of the cohomology group $H^d(\mathrm{W})$, their Hodge numbers are written as,

$$h^{d,0} = h^{d-1,1} = \cdots = h^{1,d-1} = h^{d,0} = 1 \ \ .$$

3.2 The Hermitian Metrics

Let us take a set of elements $\{\omega_l\} \in H^{l,l}(\mathrm{M})$ which can be obtained from a Kähler form $J := \omega_1 \in H^{1,1}(\mathrm{M})$. Each dimension of the primary vertical subspace $H^{l,l}(\mathrm{M})$ $(0 \leq l \leq d)$ is

given as,

$$\dim H^{l,l}(\mathrm{M}) = \begin{cases} 1 + \sum_{i=0}^{\frac{d}{2}} (-1)^i \begin{pmatrix} d+2 \\ i \end{pmatrix} \cdot \begin{pmatrix} \left(\frac{d}{2}+1-i\right) \cdot (d+1) + \frac{d}{2} \\ d+1 \end{pmatrix} & , \quad (l = \frac{d}{2} \text{ and } d \text{ is even}) \ , \\ 1 & , \quad (\text{otherwise}) \ . \end{cases}$$

But the dimension of the subspaces $H_J^{l,l}(\mathrm{M})$ obtained from a Kähler form J is written down,

$$\dim H_J^{l,l}(\mathrm{M}) = 1 \ , \quad (0 \leq l \leq d) \ .$$

From these cohomology elements $\omega_l \in H_J^{l,l}(\mathrm{M})$, we can construct physical observables $\mathcal{O}^{(l)} := \phi_A[\omega_l]$ in the $A(\mathrm{M})$-model and $\bar{\mathcal{O}}^{(\bar{l})} := \phi_{A^*}[\omega_l]$ in the $A^*(\mathrm{M})$-model. These elements describe properties of the holomorphic part and the anti-holomorphic one in the target space M. Let us take each one element from the A-model part and the A^*-model one. These two elements $\mathcal{O}^{(l)}$, $\bar{\mathcal{O}}^{(\bar{m})}$ are combined into one correlator $\mathsf{g}_{l\bar{m}} := \left\langle \bar{\mathcal{O}}^{(\bar{m})} \mathcal{O}^{(l)} \right\rangle$. These correlators are different from the topological metrics in the $A(\mathrm{M})$-model. These $\mathsf{g}_{l\bar{m}}$ are hermitian as matrices and are called the hermitian metrics.

Now we explain the definition of the expectation value in the above formula. Firstly note that the $A(\mathrm{M})$-model Lagrangian L_A can be written as,

$$L_A = \int_\Sigma d^2z \left\{ Q^{(+)} , V^{(-)} \right\} - \sqrt{-1} \int_\Sigma X^*(e) \ , \tag{4}$$

where $Q^{(+)}$ is a BRST charge of the $A(\mathrm{M})$-model and $V^{(-)}$ is defined as,

$$V^{(-)} := \sqrt{-1} g_{i\bar{\jmath}} (\rho_z^{\bar{\jmath}} \partial_{\bar{z}} X^i + \rho_{\bar{z}}^i \partial_z X^{\bar{\jmath}}) \ ,$$
$$g_{i\bar{\jmath}} \ : \text{ a metric of the Calabi-Yau d-fold M} \ .$$

Also the e is a Kähler form of M,

$$e := \sqrt{-1} g_{i\bar{\jmath}} dX^i \wedge dX^{\bar{\jmath}} \ ,$$

and the second term of L_A,

$$\int_\Sigma X^*(e) = \int_\Sigma \sqrt{-1} g_{i\bar{\jmath}} (\partial_z X^i \partial_{\bar{z}} X^{\bar{\jmath}} - \partial_{\bar{z}} X^i \partial_z X^{\bar{\jmath}}) d^2z \ ,$$

is the integral of the pullback of the Kähler form e.

Secondly the Lagrangian L_{A^*} of the $A^*(\mathrm{M})$-model can be expressed as,

$$L_{A^*} = \int_\Sigma d^2z \left\{ Q^{(-)} , V^{(+)} \right\} + \sqrt{-1} \int_\Sigma X^*(e) \ , \tag{5}$$

where $Q^{(-)}$ is a BRST charge of the $A^*(\mathrm{M})$-model and $V^{(+)}$ is defined as,

$$V^{(+)} := \sqrt{-1} g_{i\bar{\jmath}} (\bar{\rho}_{\bar{z}}^{\bar{\jmath}} \partial_z X^i + \bar{\rho}_z^i \partial_{\bar{z}} X^{\bar{\jmath}}) \ .$$

The second term in the (4)(5) has a peculiar property,

$$\int_\Sigma X^*(e) = \int_\Sigma \left\{Q_R\,,\left[\widetilde{Q}_L\,,\sqrt{-1}g_{i\bar{j}}\,\psi^i_L\psi^{\bar{j}}_R\right]\right\} + \int_\Sigma \left\{\widetilde{Q}_R\,,\left[Q_L\,,\sqrt{-1}g_{i\bar{j}}\,\psi^i_R\psi^{\bar{j}}_L\right]\right\} \ , \qquad (6)$$

where $Q_L\,,\,\widetilde{Q}_L\,,\,Q_R\,,\,\widetilde{Q}_R$ are super charges and $(\psi^i_L\,,\,\psi^{\bar{j}}_L)$, $(\psi^i_R\,,\,\psi^{\bar{j}}_R)$ are fermion pairs of the original N=2 non-linear sigma model. The operators $g_{i\bar{j}}\psi^i_L\psi^{\bar{j}}_R$ and $g_{i\bar{j}}\psi^i_R\psi^{\bar{j}}_L$ in the integrals are translated in the A(M)-model and/or A^*(M)-model after twisting,

Operators	A-model	A^*-model
$g_{i\bar{j}}\psi^i_L\psi^{\bar{j}}_R$	$g_{i\bar{j}}\chi^i\chi^{\bar{j}}$	$g_{i\bar{j}}\bar{\rho}^i_z\bar{\rho}^{\bar{j}}_{\bar{z}}$
$g_{i\bar{j}}\psi^i_R\psi^{\bar{j}}_L$	$g_{i\bar{j}}\rho^i_{\bar{z}}\rho^{\bar{j}}_z$	$g_{i\bar{j}}\bar{\chi}^i\bar{\chi}^{\bar{j}}$

For the A-model, the operator $\mathcal{O}^{(1)} := g_{i\bar{j}}\chi^i\chi^{\bar{j}}$ is a BRST observable associated to the Kähler form e, but the other operator $g_{i\bar{j}}\rho^i_{\bar{z}}\rho^{\bar{j}}_z$ is not a BRST observable and not a physical one. However the second term $\left\{\widetilde{Q}_R\,,\left[Q_L\,,\sqrt{-1}g_{i\bar{j}}\,\rho^i_{\bar{z}}\rho^{\bar{j}}_z\right]\right\}$ in (6) works trivially in physical situation except for the cases when one must consider some contributions from the boundary of the moduli space because $Q^{(+)} = Q_L + \widetilde{Q}_R$ is a BRST charge of A-model.

Conversely the operator $\bar{\mathcal{O}}^{(\bar{1})} := g_{i\bar{j}}\bar{\chi}^i\bar{\chi}^{\bar{j}}$ turns out to be a physical one associated with the Kähler form e and the other operator $g_{i\bar{j}}\bar{\rho}^i_z\bar{\rho}^{\bar{j}}_{\bar{z}}$ becomes unphysical for the A^*-model case. In similar to the A-model case, the first term $\left\{Q_R\,,\left[\widetilde{Q}_L\,,\sqrt{-1}g_{i\bar{j}}\,\bar{\rho}^i_z\bar{\rho}^{\bar{j}}_{\bar{z}}\right]\right\}$ in (6) works trivially in usual physical situations because $Q^{(-)} = \widetilde{Q}_L + Q_R$ is a BRST operator of A^*-model.

Next we think about the meaning of the twisting. This twisting method can also be understood in the Lagrangian formalism. To begin with, we consider a Lagrangian L_0 (1) of the original N=2 non-linear sigma model. The (quasi)-topological field theories (the A-model and the A^*-model) are obtained by introducing a fermion number current J (one-form on the Riemann surface Σ),

$$L_A = L_0 + \int_\Sigma \mathsf{J}\wedge\left(\frac{i}{2}\varsigma\right) \ , \ (A-\text{model}) \ ,$$
$$L_{A^*} = L_0 + \int_\Sigma \mathsf{J}\wedge\left(\frac{-i}{2}\varsigma\right) \ , \ (A^*-\text{model}) \ ,$$

which is coupled to the spin connection ς on the Riemann surface Σ. By this recipe, the spins of fermions are changed by an amount depending on their U(1) charges and then fermions come to take values not on spin bundles but on (anti)-canonical bundles. The objects we want to obtain are the hermitian metrics $\mathsf{g}_{l\bar{m}} := \left\langle\bar{\mathcal{O}}^{(\bar{m})}\mathcal{O}^{(l)}\right\rangle$. Because operators $\mathcal{O}^{(l)}$, $\bar{\mathcal{O}}^{(\bar{m})}$ are observables of the A-model, A^*-model respectively, it is necessary to merge these two models into one theory. In order to fulfill our purpose, we take a genus 0 Riemann surface as a world sheet Σ constructed from two hemi-spheres Σ_L and Σ_R,

$$\Sigma = \Sigma_L \cup \Sigma_R \ ,$$
$$\Sigma_L \cap \Sigma_R \cong S^1 \ ,$$

where subscripts L and R stand for the left, right part of Σ respectively. We put a topological theory on Σ_L, an anti-topological one on Σ_R and connect them smoothly on Σ. A Lagrangian of the resulting theory is defined as,

$$\begin{aligned} L(t\,,\,\bar{t}) = L_0 + \int_\Sigma \mathsf{J} \wedge \mathsf{A} \\ +t\int \left\{Q_R\,,\left[\widetilde{Q}_L\,,\sqrt{-1}g_{i\bar{j}}\,\psi^i_L\psi^{\bar{j}}_R\right]\right\} \\ +\bar{t}\int \left\{\widetilde{Q}_R\,,\left[Q_L\,,\sqrt{-1}g_{i\bar{j}}\,\psi^i_R\psi^{\bar{j}}_L\right]\right\} \quad , \end{aligned} \tag{7}$$

where A is a U(1) gauge connection on Σ. This A becomes $\frac{i}{2}\varsigma$ in the far-right and turns into $\frac{-i}{2}\varsigma$ in the far-left on Σ smoothly. The third and fourth terms on the right hand side in (7) are perturbation terms in order to look into the response for the marginal perburbations associated with the Kähler form e. Using this Lagrangian, we can write a definition of the hermitian metrics,

$$\begin{aligned} \mathsf{g}_{l\bar{m}} &= \left\langle \bar{\mathcal{O}}^{(\bar{m})}\mathcal{O}^{(l)} \right\rangle \\ &:= \int \mathcal{D}[X,\psi]\,\bar{\mathcal{O}}^{(\bar{m})}\mathcal{O}^{(l)}e^{-L(t\,,\bar{t})} \quad . \end{aligned}$$

By analyzing them in the operator formalism, we obtain an equation (AA^*-equation) for these correlators $\mathsf{g}_{l\bar{m}}$,

$$\begin{aligned} &\partial_{\bar{t}}\left(\mathsf{g}\partial_t\mathsf{g}^{-1}\right) = \left[C_t\,,\mathsf{g}\bar{C}_{\bar{t}}\mathsf{g}^{-1}\right] \quad , \\ &(C_t)_{lm} = \left\langle \mathcal{O}^{(m)}\middle|\mathcal{O}^{(1)}\middle|\mathcal{O}^{(l)}\right\rangle \quad , \\ &\left(\bar{C}_{\bar{t}}\right)_{\bar{l}\bar{m}} = \left\langle \bar{\mathcal{O}}^{(\bar{m})}\middle|\bar{\mathcal{O}}^{(\bar{1})}\middle|\bar{\mathcal{O}}^{(\bar{l})}\right\rangle \quad , \end{aligned} \tag{8}$$

where (C_t), $\left(\bar{C}_{\bar{t}}\right)$ are three point functions containing operators $\mathcal{O}^{(1)}$, $\bar{\mathcal{O}}^{(\bar{1})}$ for the A(M)-model, A^*(M)-model respectively. (The derivation of this equation is explained in the Appendix A).

The hermitian metrics are determined by three point correlators of the A (A^*)-models. The author obtained these couplings of the A(M)-model for the Calabi-Yau d-fold [13]. (A short review of the derivation of these correlators are explained in the Appendix B). In the next section, we apply this formula to the Calabi-Yau d-fold case (2) in order to obtain expressions of the metrics.

4 Application of the AA^*-Equation

In this section, we apply the AA^*-equation to the Calabi-Yau d-fold M (2) and analyze properties of the hermitian metrics $\mathsf{g}_{l\bar{m}}$. Also by using these results, a genus one partition function can be obtained.

4.1 The Hermitian Metrics of the Calabi-Yau d-Fold

When we take the set of the observables $\{\mathcal{O}^{(l)}\}$, $\left(\mathcal{O}^{(l)} \in H_J^{l,l}(\mathrm{M})\ ;\ (l = 0, 1, \cdots, d)\right)$ for the d-fold M as a basis, the structure constant of the operator product between $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(l)}$ can be represented as a matrix C_t [13],

$$C_t := \begin{pmatrix} 0 & \mathcal{K}_0 & & & & & O \\ & 0 & \mathcal{K}_1 & & & & \\ & & 0 & \mathcal{K}_2 & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & \mathcal{K}_{d-2} & \\ & & & & & 0 & \mathcal{K}_{d-1} \\ O & & & & & & 0 \end{pmatrix} . \tag{9}$$

Let us apply the AA^*-equation in the previous section. The deformation parameter t coupled to the charge one field $\mathcal{O}^{(1)}$ associated with a Kähler form e is the coordinate on the complexified Kähler moduli space of the A(M)-model. In our case, we can take the hermitian metrics diagonally,

$$\mathsf{g} := \mathrm{diag}(\, e^{q_0}\ e^{q_1}\ \cdots\ e^{q_{d-1}}\ e^{q_d}\,)\ .$$

Substituting this matrix into the AA^*-equation, we obtain a set of differential equations,

$$\begin{aligned} &\partial_{\bar{t}}\partial_t q_0 + |\mathcal{K}_0|^2 e^{q_1 - q_0} = 0\ , \\ &\partial_{\bar{t}}\partial_t q_l + |\mathcal{K}_l|^2 e^{q_{l+1} - q_l} - |\mathcal{K}_{l-1}|^2 e^{q_l - q_{l-1}} = 0\ ,\ \ (l = 1, 2, \cdots, d-1)\ , \\ &\partial_{\bar{t}}\partial_t q_d - |\mathcal{K}_{d-1}|^2 e^{q_d - q_{d-1}} = 0\ , \end{aligned} \tag{10}$$

This set can be rewritten in a compact form by introducing new variables φ_l,

$$\begin{aligned} &\varphi_l := q_l - q_{l-1} + \log(\mathcal{K}_{l-1}\bar{\mathcal{K}}_{l-1}) \quad (l = 1, 2, \cdots, d)\ , \\ &\partial_{\bar{t}}\partial_t \varphi_l = \sum_{m=1}^{d} K_{lm} e^{\varphi_m} \quad (l = 1, 2, \cdots, d-1, d)\ , \end{aligned}$$

where the coefficients K_{lm} are given by changing the sign of each component of the Cartan matrix of the Lie algebra A_d. The above system is the A-type Toda equation system. We make one remark here; The AA^*-equation system in the CP^{d+1} model is that of the affine A-type Toda theory. In our Calabi-Yau d-fold case, the system is that of non-affine A-type Toda theory. This difference stems from the property of the first Chern classes $c_1(\mathrm{M})$ of the Kähler manifolds M. The virtual dimension (the ghost number anomaly) of the correlators in the A(M)-model can be written,

$$(\dim_{\mathbf{C}} M) \cdot (1 - g) + \int_{\Sigma} X^* c_1(M)\ , \tag{11}$$

where g is a genus of a Riemann surface Σ. The second term in the above formula depends on the degree of the map X. For each fixed degree n of the map, there exists one topological selection rule. For the CP^{d+1} case, this second term is $(d+2)\,n$. Also we deform some topological field theory by adding only operators associated to a Kähler form. Then the degree of the observables are conserved in each fusion. For the CP^{d+1} case, the next fusion is non-vanishing generally,

$$\mathcal{O}^{(1)} \cdot \mathcal{O}^{(d+1)} \neq 0 \ ,$$

and leads to an affine Toda theory. On the other hand, the second term vanishes for the manifolds with the vanishing first Chern classes (Calabi-Yau manifolds). In these cases, the following fusion leads to zero,

$$\mathcal{O}^{(1)} \cdot \mathcal{O}^{(d)} = 0 \ .$$

It is this reason that the AA^*-equations for the CP^{d+1} model is affine Toda, but that for Calabi-Yau manifolds are non-affine Toda.

Return to the analysis of the Toda equation in our case. We solve the equation explicitly. Firstly let us introduce a set of Grassmann variables ξ^a $(a = 0, 1, \cdots, d)$ and derivatives $\dfrac{\partial}{\partial \xi^b}$ $(b = 0, 1, \cdots, d)$, which satisfy the anti-commutation relations,

$$\begin{aligned} &\left\{ \xi^a \, , \xi^b \right\} = \left\{ \frac{\partial}{\partial \xi^a} \, , \frac{\partial}{\partial \xi^b} \right\} = 0 \ , \\ &\left\{ \xi^a \, , \frac{\partial}{\partial \xi^b} \right\} = \delta_{ab} \ . \end{aligned}$$

Secondly we define the functions ζ^+, ζ^- by using arbitrary functions (ζ^+_a, ζ^-_b),

$$\zeta^+(t) := \sum_{a=0}^{d} \zeta^+_a(t) \cdot \xi^a \ , \tag{12}$$

$$\zeta^-(\bar{t}) := \sum_{b=0}^{d} \zeta^-_b(\bar{t}) \cdot \frac{\partial}{\partial \xi^b} \ , \tag{13}$$

where $\{\zeta^+_a\}$ ($\{\zeta^-_b\}$) are holomorphic (anti-holomorphic) functions with respect to the variable t. The set of solutions for e^{q_n} is obtained,

$$\begin{aligned} e^{q_n} &= \frac{\mathsf{H}_{n+1}}{\mathsf{H}_n} \cdot \frac{B_0}{\left|\mathcal{K}_0 \mathcal{K}_1 \cdots \mathcal{K}_{n-1}\right|^2} \ , \\ \mathsf{H}_n &= \sum_{0 \le i_1 < \cdots < i_n \le d} \Big| (\zeta^+_{i_1})(\zeta^+_{i_2}) \cdots (\zeta^+_{i_n}) \Big|_t \times \Big| (\zeta^-_{i_1})(\zeta^-_{i_2}) \cdots (\zeta^-_{i_n}) \Big|_{\bar{t}} \ , \ \ (n = 1, 2, \cdots) \ , \\ H_0 &:= 1 \ , \end{aligned}$$

with one relation,

$$
\begin{aligned}
1 \;=\;& (\zeta^+)(\zeta^+)'(\zeta^+)''\cdots(\zeta^+)^{(d)} \\
&\times(\zeta^-)(\zeta^-)'(\zeta^-)''\cdots(\zeta^-)^{(d)} \ , \\
&\quad \text{where } \ (\zeta^+)^{(n)} = \partial_t^n \zeta^+ \ , \ \ (\zeta^-)^{(n)} = \bar{\partial}_{\bar{t}}^n \zeta^- \ ,
\end{aligned}
\tag{14}
$$

where we used a notation to save the space,

$$
|\, a_1\, a_2\, \cdots\, a_m \,|_x := \begin{vmatrix} a_1 & a_2 & \cdots & a_m \\ \partial_x a_1 & \partial_x a_2 & \cdots & \partial_x a_m \\ \vdots & \vdots & & \vdots \\ \partial_x^{m-1} a_1 & \partial_x^{m-1} a_2 & \cdots & \partial_x^{m-1} a_m \end{vmatrix} \ .
$$

Also the B_0 is some function represented as products of some pure holomorphic functions and anti-holomorphic ones, which are determined from the boundary condition. We use a set of functions as candidate for the general solution (10),

$$
\zeta_a^+ = \left(\varpi_0^{-1} \cdot \prod_{j=0}^{d-1} \mathcal{K}_j^{-\frac{d-j}{d+1}} \right) \cdot \varpi_a \ , \ \ (a = 0, 1, \cdots, d) \ , \tag{15}
$$

$$
\zeta_b^- = \left(\bar{\varpi}_0^{-1} \cdot \prod_{j=0}^{d-1} \bar{\mathcal{K}}_j^{-\frac{d-j}{d+1}} \right) \cdot \bar{\varpi}_{d-b} \cdot (-1)^{d-b} \ , \ \ (b = 0, 1, \cdots, d) \ , \tag{16}
$$

where the $\mathcal{K}_j$ are the three point couplings written in (9) and the $\bar{\mathcal{K}}_j$ are their complex conjugate. The $\varpi_a(z)$ are the solutions of the Picard-Fuchs equation and written by using the Schur polynomials,

$$
\begin{aligned}
&\varpi_a(z) = \varpi_0(z) \cdot S_a(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_a) \ , \ \ (a = 0, 1, \cdots, d) \ , \\
&\hat{\varpi}_0(z\,;\,\rho) := \sum_{n=0}^{\infty} \frac{\Gamma(N(n+\rho)+1)}{\Gamma(N\rho+1)} \cdot \left[\frac{\Gamma(\rho+1)}{\Gamma(n+\rho+1)} \right]^N \cdot (N\psi)^{-N(n+\rho)} \ , \\
&\tilde{x}_m := \frac{1}{m!} \mathcal{D}_\rho^m \log \hat{\varpi}_0(z\,;\,\rho)\big|_{\rho=0} \ , \ \ \varpi_0(z) := \hat{\varpi}_0(z\,;\,\rho)\Big|_{\rho=0} \ , \\
&z := (N\psi)^{-N} \ ,
\end{aligned}
$$

Also the $\bar{\varpi}_a(\bar{z})$ are complex conjugate of the $\varpi_a(z)$. These functions satisfy a condition,

$$
\mathsf{H}_{d+1} = \Big|(\zeta_0^+)(\zeta_1^+)\cdots(\zeta_d^+)\Big|_t \times \Big|(\zeta_0^-)(\zeta_1^-)\cdots(\zeta_d^-)\Big|_{\bar{t}} = 1 \ , \tag{17}
$$

where we used a relation,

$$
\begin{aligned}
&|\, \omega_0\, \omega_1\, \cdots\, \omega_d \,|_t = (\omega_0)^{d+1} (\mathcal{K}_{d-1})^1 (\mathcal{K}_{d-2})^2 \cdots (\mathcal{K}_1)^{d-1} (\mathcal{K}_0)^d \ , \\
&\omega_a := \frac{\varpi_a}{\varpi_0} \ \ (a = 0, 1, \cdots, d) \ .
\end{aligned}
$$

In addition, we choose the function B_0,

$$B_0 = (\varpi_0 \bar{\varpi}_0) \cdot \sum_{j=0}^{d-1} (\mathcal{K}_j \bar{\mathcal{K}}_j)^{-\frac{d-j}{d+1}} \ .$$

Under this setup, we can write down solutions e^{q_n},

$$\begin{aligned} \exp(q_n) \ &= \ (\varpi_0 \cdot \bar{\varpi}_0) \cdot \frac{\displaystyle\sum_{0 \le i_1 < \cdots < i_{n+1} \le d} \Big| \omega_{i_1}\, \omega_{i_2}\, \cdots\, \omega_{i_{n+1}} \Big|_t \cdot \Big| \tilde{\omega}_{i_1}\, \tilde{\omega}_{i_2}\, \cdots\, \tilde{\omega}_{i_{n+1}} \Big|_{\bar{t}}}{\displaystyle\sum_{0 \le j_1 < \cdots < j_n \le d} \big| \omega_{j_1}\, \omega_{j_2}\, \cdots\, \omega_{j_n} \big|_t \cdot \big| \tilde{\omega}_{j_1}\, \tilde{\omega}_{j_2}\, \cdots\, \tilde{\omega}_{j_n} \big|_{\bar{t}}} \\ &\quad \times \frac{1}{\left|\mathcal{K}_0 \mathcal{K}_1 \cdots \mathcal{K}_{n-1}\right|^2} \ , \\ &\quad \omega_a := \frac{\varpi_a}{\varpi_0} \ , \ \ \tilde{\omega}_a := \frac{\bar{\varpi}_{d-a}}{\bar{\varpi}_0} \cdot (-1)^{d-a} \ . \end{aligned}$$

(These results remind us of the structure of the W-geometry of the Toda systems [24]). Also the product of the couplings can be expressed as,

$$\left|\mathcal{K}_0 \mathcal{K}_1 \cdots \mathcal{K}_{n-1}\right|^2 = \frac{\big| \omega_0\, \omega_1\, \cdots\, \omega_n \big|_t \cdot \big| \bar{\omega}_0\, \bar{\omega}_1\, \cdots\, \bar{\omega}_n \big|_{\bar{t}}}{\big| \omega_0\, \omega_1\, \cdots\, \omega_{n-1} \big|_t \cdot \big| \bar{\omega}_0\, \bar{\omega}_1\, \cdots\, \bar{\omega}_{n-1} \big|_{\bar{t}}} \ . \tag{18}$$

Let us make two remarks: Firstly each summand in the solution e^{q_n} behaves in the large radius limit $\mathrm{Im}\, t \to \infty$ as,

$$\begin{aligned} \Big| \omega_{i_1}\ \omega_{i_2}\ \cdots\ \omega_{i_{n+1}} \Big|_t &\sim t^{i_1 + i_2 + \cdots + i_{n+1} - \frac{1}{2}n(n+1)} \ , \\ \Big| \tilde{\omega}_{i_1}\ \tilde{\omega}_{i_2}\ \cdots\ \tilde{\omega}_{i_{n+1}} \Big|_{\bar{t}} &\sim \bar{t}^{(d-i_1)+(d-i_2)+(d-i_{n+1}) - \frac{1}{2}n(n+1)} \ . \end{aligned}$$

On the other hand, the product of the couplings (18) tends to constant classical one in this limit. When one scales the parameters $(t\,,\,\bar{t})$ as $\lambda\,(t\,,\,\bar{t})$ in this limit by using a real positive parameters λ, each solution e^{q_n} behaves as,

$$e^{q_n} \sim \lambda^{-2n+d} \ .$$

We identify the scale dimension of the n-th component of the set of solutions e^{q_n} with n which is the same as that of the hermitian metric $g_{n\bar{n}} = \left\langle \bar{\mathcal{O}}^{(\bar{n})} \middle| \mathcal{O}^{(n)} \right\rangle$. (The extra scaling exponent d comes from the axial U(1) anomaly due to the axial couplings between the U(1) gauge connection A and the fermion number current J). As a second remark, we consider the rescaling of the periods $\{\varpi_a\}$. When we rescale $(\varpi_a, \bar{\varpi}_b)$ by multiplying arbitrary holomorphic (anti-holomorphic) functions $f(t)$, $\bar{f}(\bar{t})$ respectively,

$$(\varpi_a, \bar{\varpi}_b) \to (f\, \varpi_a, \bar{f}\, \bar{\varpi}_b) \ ,$$

the solutions e^{q_n} transform into the forms,

$$
\begin{aligned}
& e^{q_n} \rightarrow f(t)\bar{f}(\bar{t})e^{q_n} \ , \\
& \text{i.e. } q_n \rightarrow q_n + \log f(t) + \log \bar{f}(\bar{t}) \ .
\end{aligned}
$$

This means that the q_n's are not functions but sections of holomorphic line bundle $\mathcal{L}$ over the A-model moduli space. Next let us investigate properties of the metrics. Firstly we take the e^{q_0}. It is written by a simple calculation,

$$
\begin{aligned}
& e^{q_0} = (\varpi_0 \bar{\varpi}_0) \cdot S_d(\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_d) \ , \\
& \tilde{z}_n := \tilde{x}_n + (-1)^n \cdot \bar{\tilde{x}}_n \ , \ \ \tilde{x}_m := \frac{1}{m!}\mathcal{D}_\rho^m \log \hat{\varpi}_0(z\,;\,\rho)\big|_{\rho=0} \ ,
\end{aligned}
$$

where the S_d is a Schur function. This corresponds to the two point function of charge zero operators,

$$
e^{q_0} = \mathsf{g}_{0\bar{0}} = \left\langle \bar{\mathcal{O}}^{(\bar{0})} \middle| \mathcal{O}^{(0)} \right\rangle \ .
$$

Generally for restricted Kähler manifolds, this correlator can be represented by using the Kähler potential $\mathcal{K}$,

$$
g_{0\bar{0}} = e^{-\mathcal{K}} \ .
$$

From this information, we may put $q_0 = -\mathcal{K}$. These have a characteristic property,

$$
\begin{aligned}
& \mathcal{K} \rightarrow \mathcal{K} - \log f(t) - \log \bar{f}(\bar{t}) \ , \\
& (\varpi_a, \bar{\varpi}_b) \rightarrow (f\,\varpi_a, \bar{f}\,\bar{\varpi}_b) \ .
\end{aligned}
$$

Several explicit formulae are collected in the Appendix C. Secondly we study the e^{q_1}. From the Toda equation (10), the e^{q_1} is written,

$$
\begin{aligned}
e^{q_1} &= e^{q_0} \cdot (-\partial_{\bar{t}}\partial_t q_0) \\
&= e^{q_0} \cdot (\partial_{\bar{t}}\partial_t \mathcal{K}) \ .
\end{aligned}
$$

The above derivative of the Kähler potential $\partial_{\bar{t}}\partial_t\mathcal{K}$ is known as the Zamolodchikov metric $G_{t\bar{t}} = \partial_{\bar{t}}\partial_t\mathcal{K}$ in general terms. A relation between this metric $G_{t\bar{t}}$ and the hermitian metric is written as,

$$
G_{t\bar{t}} = \frac{\mathsf{g}_{1\bar{1}}}{\mathsf{g}_{0\bar{0}}} \ .
$$

From this consideration, we identify e^{q_1} with $\mathsf{g}_{1\bar{1}}$,

$$
e^{q_1} = \mathsf{g}_{1\bar{1}} = \left\langle \bar{\mathcal{O}}^{(\bar{1})} \middle| \mathcal{O}^{(1)} \right\rangle \ .
$$

We make a conjecture about the interpretation of the other solutions.
(Conjecture)

The solutions e^{q_n} of the Toda equation (10) are identified with the hermitian metrics on the complexified Kähler moduli space,

$$\oplus_p T^p(M\,,\,\bigwedge^p \bar{T}^* M)\otimes \mathbf{C}$$

of the Calabi-Yau A(M) model.
In the rest of this section, a genus one partition function is analyzed by using the results obtained here.

4.2 The Genus One Partition Function

A genus one partition function F_1 is described by the following new index [14, 25],

$$\mathsf{F}_1 = \frac{1}{2}\cdot\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\mathrm{Tr}(-1)^F F_L F_R q^{L_0}\bar{q}^{\bar{L}_0}\ ,$$
$$F := F_L - F_R\ ,\ q := e^{2\pi i\tau}\ ,\ \bar{q} := e^{-2\pi i\bar{\tau}}\ ,$$

where the F_L, F_R are left, right fermion number operators. (Insertion of these operators adjusts the fermion zero modes). Also the trace in the above formula is over the Ramond sector ground states and the integral is calculated over the fundamental region $\mathcal{F}$ of the world sheet torus with a modulus parameter $\tau=\tau_1+i\tau_2$. This formula is rewritten by the analysis in the operator formalism,

$$\partial_{\bar{\jmath}}\partial_i\mathsf{F}_1 = \frac{1}{2}\mathrm{Tr}\left\{(-1)^F C_i\bar{C}_{\bar{\jmath}}\right\} - \frac{1}{24}G_{i\bar{\jmath}}\mathrm{Tr}(-1)^F\ , \tag{19}$$

where subscripts "i" "$\bar{\jmath}$" represent marginal operators ϕ_i, $\bar{\phi}_{\bar{\jmath}}$ on the Ramond ground states. In our situation, the "i" "$\bar{\jmath}$" are associated to A-model, A^*-model operators $\phi_A[e_i]$, $\phi_{A^*}[e_j]$ constructed from Kähler forms e_i, e_j of M respectively. The symbol C_i, $\bar{C}_{\bar{\jmath}}$ are structure constants associated to the above marginal operators and the trace is over the Ramond ground states. In the second term in the right hand side, the $G_{i\bar{\jmath}}$ is Zamolodchikov metrics of the Calabi-Yau Kähler moduli space. Also the $\mathrm{Tr}(-1)^F$ is equal to a Euler number χ(M) of the Calabi-Yau d-fold M. The derivatives ∂_i, $\partial_{\bar{\jmath}}$ in the left hand side are partial derivatives with respect to marginal coordinates t_i, $\bar{t}_{\bar{\jmath}}$ associated to operators $\phi_A[e_i]$, $\phi_{A^*}[e_j]$ respectively.

Furthermore the equation can be rewritten as,

$$\partial_i\partial_{\bar{\jmath}}\mathsf{F}_1 = \frac{1}{2}\partial_i\partial_{\bar{\jmath}}\log P\ , \tag{20}$$

$$\log P := \sum_{p,q}(-1)^{p-q}\frac{p+q}{2}\mathrm{Tr}_{p,q}\left(\log \mathsf{g}\right) - \frac{\mathcal{K}}{12}\mathrm{Tr}(-1)^F\ , \tag{21}$$

where the variables p,q in the summand are left, right Ramond U(1) charges and also equal to the ghost numbers of the A-model, A^*-model operators. The "g" in the trace are hermitian metrics $\mathsf{g}_{p\bar{q}} = \left\langle\bar{\mathcal{O}}^{(\bar{q})}\mathcal{O}^{(p)}\right\rangle$ of the Kähler moduli space. Especially the $\mathcal{K}$ is a Kähler potential of the moduli space and is represented as $\mathcal{K} = -\log \mathsf{g}_{0\bar{0}}$.

4.3 Application to the Calabi-Yau d-Fold

In this subsection, we apply the recipe in the previous subsection to the Calabi-Yau d-fold (2). For this Calabi-Yau d-fold ($c = 3d$), the Ramond U(1) charges q_R are written,

$$q_R = l - \frac{d}{2} \ , \ \ (l = 0, 1, \cdots, d) \ .$$

Especially their conformal weights h_R is constant, $h_R = \frac{c}{24} = \frac{d}{8}$. The corresponding weight h_{NS} and U(1) charges q_{NS} in the Neveu-Schwarz sector are obtained by the spectral flow,

$$\begin{aligned} h_{NS} &= h_R + \frac{1}{2} q_R + \frac{d}{8} = \frac{l}{2} \ , \\ q_{NS} &= q_R + \frac{d}{2} = l \ , \ \ (l = 0, 1, \cdots, d) \ . \end{aligned}$$

Note that the factor $(-1)^{p-q} \cdot \dfrac{p+q}{2}$ is calculated in the case $p = l - \frac{d}{2}$, $q = m - \frac{d}{2}$ as,

$$\begin{aligned} (-1)^{p-q} \cdot \frac{p+q}{2} &= \frac{1}{2}(r-d) \cdot (-1)^r \ , \\ r := l + m \ &, \ \ (l = 0, 1, \cdots, d \, ; \, m = 0, 1, \cdots, d) \ . \end{aligned}$$

From this consideration, it turns out to be that the primary horizontal subspace of the de Rham cohomology of M,

$$H^d(\mathrm{M}) := \oplus_{s=0}^{d} H^{d-s,s}(\mathrm{M}) \ ,$$

does not contribute to the F_1. Because the Hodge numbers are given as (3) in our model (2), $\log P$ is obtained,

$$\log P = -\frac{\chi}{12} \cdot \mathcal{K} + \sum_{s=0}^{d} \left(s - \frac{d}{2} \right) \cdot \log(\mathsf{g}_{s\bar{s}}) \ . \tag{22}$$

By using a reality condition of the hermitian metrics,

$$\mathsf{g}_{d-s,\overline{d-s}} = \frac{1}{\mathsf{g}_{s\bar{s}}} \ , \tag{23}$$

we obtain a formula,

$$\log P = \left(-\frac{\chi}{12} + \left[\frac{d+1}{2} \right]_G \left[\frac{d+2}{2} \right]_G \right) \mathcal{K} - \sum_{s=1}^{\left[\frac{d-1}{2}\right]_G} (d - 2s) \log G_{s\bar{s}} \ , \tag{24}$$

$$G_{s\bar{s}} := \frac{\mathsf{g}_{s\bar{s}}}{\mathsf{g}_{0\bar{0}}} \ , \ \ (s = 0, 1, \cdots, d) \ . \tag{25}$$

We define a symbol $[x]_G$ as a unique maximal integer which is not greater than x. Thus the partition function F_1 is written as,

$$\mathsf{F}_1 = \frac{1}{2}\cdot\log\left[\exp\left\{\left(-\frac{\chi}{12}+\left[\frac{d+1}{2}\right]_G\left[\frac{d+2}{2}\right]_G\right)\mathcal{K}\right\}\times\prod_{s=1}^{\left[\frac{d-1}{2}\right]_G}(G_{s\bar{s}})^{-(d-2s)}\cdot|f|^2\right] \ , \tag{26}$$

where f is an unknown holomorphic function determined by a boundary condition. Now we use the mirror symmetry of the d-fold M and its partner W. Recall that W is constructed as a orbifold from M divided by the maximally invariant discrete group $(\mathbf{Z}_{d+2})^{d+1}$. The complex moduli space of W has singularities at points $\psi = 0, \tilde{\alpha}^a$, and ∞ ($a = 0, 1, \cdots, d+1\,;\ \tilde{\alpha} := e^{\frac{2\pi i}{d+2}}$). Because the behaviour of the function f is controlled by these singular points, we make a conjecture about f,

$$f = \psi^{\alpha}(1-\psi^{d+2})^{\beta} \ , \tag{27}$$

with some constant numbers α and β. Thus we write a genus one partition function of W as,

$$\begin{aligned}
\mathsf{F}_1(\psi) &= \frac{1}{2}\cdot\log\left[\exp\left\{\left(-\frac{\chi}{12}+\left[\frac{d+1}{2}\right]_G\left[\frac{d+2}{2}\right]_G\right)\mathcal{K}\right\}\right. \\
&\qquad \left.\times\prod_{s=1}^{\left[\frac{d-1}{2}\right]_G}(G_{s\bar{s}}(\psi))^{-(d-2s)}\cdot\left|\psi^{\alpha}(1-\psi^{d+2})^{\beta}\right|^2\right] \ , \qquad (28) \\
\mathcal{K} &= -\log \mathsf{g}_{0\bar{0}} \ , \\
\mathsf{g}_{0\bar{0}} &= (\varpi_0\bar{\varpi}_0)\cdot S_d(\tilde{z}_1\,,\,\tilde{z}_2\,,\,\cdots\,,\,\tilde{z}_d) \ , \qquad (29) \\
G_{n\bar{n}}(\psi) &= e^{q_n-q_0} \\
&= \frac{1}{S_d(\tilde{z})}\cdot\frac{\displaystyle\sum_{0\le i_1<\cdots<i_{n+1}\le d}\left|\,\omega_{i_1}\,\omega_{i_2}\,\cdots\,\omega_{i_{n+1}}\right|_{\psi}\cdot\left|\tilde{\omega}_{i_1}\,\tilde{\omega}_{i_2}\,\cdots\,\tilde{\omega}_{i_{n+1}}\right|_{\bar{\psi}}}{\displaystyle\sum_{0\le j_1<\cdots<j_n\le d}\left|\,\omega_{j_1}\,\omega_{j_2}\,\cdots\,\omega_{j_n}\right|_{\psi}\cdot\left|\tilde{\omega}_{j_1}\,\tilde{\omega}_{j_2}\,\cdots\,\tilde{\omega}_{j_n}\right|_{\bar{\psi}}} \\
&\qquad \times\frac{1}{|\mathcal{K}_0\mathcal{K}_1\cdots\mathcal{K}_{n-1}|^2} \ . \qquad (30)
\end{aligned}$$

Because the partition function is a zero point function, the corresponding one $\mathsf{F}_1(t)$ in the A(M)-model can be obtained only by changing parameters from ψ to $t(\psi) := \tilde{x}_1$,

$$\mathsf{F}_1(t) = \mathsf{F}_1(\psi(t)) \ . \tag{31}$$

All we have to do is to determine the unknown constant numbers α and β. To achieve this purpose, we examine asymptotic behaviours of the F_1 in some limits.

Firstly in the large radius limit, the F_1 becomes,

$$\begin{aligned} \mathsf{F}_1\Big|_{(t,\bar{t})\to\infty} &= \frac{-1}{24}(t+\bar{t})\int_M e\wedge c_{d-1} \\ &= (t+\bar{t})\cdot\frac{-1}{24}\cdot\left\{\frac{1}{N^2}\left[1-(1-N)^N\right]+\frac{1}{2}(N-2)(N+1)\right\} \ , \\ N &:= d+2 \ . \end{aligned} \tag{32}$$

Secondly we take an asymmetrical limit $\bar{t}\to\infty$ while keeping t fixed. In this limit, the normalized hermitian metrics $G_{n\bar{n}}$ behaves as,

$$\begin{aligned} \log G_{n\bar{n}}(t) &\sim -\log(S_d)-\log|\mathcal{K}_0\mathcal{K}_1\cdots\mathcal{K}_{n-1}|^2 \\ &\quad +\log|\mathcal{K}_0\mathcal{K}_1\cdots\mathcal{K}_{n-1}|+\log\frac{|\tilde{\omega}_0\tilde{\omega}_1\cdots\tilde{\omega}_n|_{\bar{t}}}{|\tilde{\omega}_0\tilde{\omega}_1\cdots\tilde{\omega}_{n-1}|_{\bar{t}}} \\ &\quad =-\log(S_d)+(\text{anti-holomorphic parts})+\mathcal{O}(\bar{t}^{-1}) \ . \end{aligned} \tag{33}$$

Now the Schur polynomial S_d is written down,

$$S_d=\frac{1}{d!}\cdot(t-\bar{t})^d+\frac{1}{(d-2)!}\cdot(t-\bar{t})^{d-2}\cdot\left(D_\rho t+\overline{D_\rho t}\right)\Big|_{\rho=0}+\cdots \ ,$$
$$\text{with } D_\rho t:=\left(\frac{1}{2\pi i}\cdot\frac{\partial}{\partial\rho}\right)^2\log\hat{\varpi}_0(z\,;\,\rho) \ ,$$

and then,

$$\log S_d\sim\log(\bar{t})^d+\mathcal{O}(\bar{t}^{-1}) \ , \ (\bar{t}\to\infty) \ .$$

Thus the metric is reduced to an asymptotic form,

$$\log G_{n\bar{n}}(t)\sim(\text{anti-holomorphic parts})+\mathcal{O}(\bar{t}^{-1}) \ . \tag{34}$$

Similarly the Kähler potential $\mathcal{K}$ behaves as,

$$\begin{aligned} \mathcal{K} &= -\log \mathsf{g}_{0\bar{0}} \\ &\sim -\log\varpi_0+(\text{anti-holomorphic parts})+\mathcal{O}(\bar{t}^{-1}) \ , \end{aligned}$$

in the limit $\bar{t}\to\infty$. By using a transformation property,

$$G_{n\bar{n}}(\psi)=\left(\frac{\partial t}{\partial\psi}\right)^n\overline{\left(\frac{\partial t}{\partial\psi}\right)}^n\cdot G_{n\bar{n}}(t) \ , \tag{35}$$

we obtain an equation,

$$\mathsf{F}_1(\bar{t}\to\infty)=\frac{1}{2}\cdot\log\left[(\varpi_0)^{-v}\left(\frac{\partial\psi}{\partial t}\right)^u\cdot\psi^\alpha(1-\psi^N)^\beta\right] \ , \tag{36}$$

$$\chi = \frac{1}{N} \cdot \left[(1-N)^N - 1 + N^2 \right] \ \ , \ \ \text{(Euler number of M)} \ \ , \tag{37}$$

$$v := -\frac{\chi}{12} + \left[\frac{d+1}{2}\right]_G \left[\frac{d+2}{2}\right]_G \ , \tag{38}$$

$$u := \begin{cases} \frac{1}{6}\left[\frac{d-1}{2}\right]_G \left[\frac{d+1}{2}\right]_G \cdot d & (d\,;\,\text{odd}) \\ \frac{1}{6}\left[\frac{d-1}{2}\right]_G \left[\frac{d+1}{2}\right]_G \cdot (d+2) & (d\,;\,\text{even}) \end{cases} \ . \tag{39}$$

By postulating the regularity at the singular points $\psi = 0$ and ∞ of the complex moduli space of W, we obtain the constants α and β,

$$\alpha = v \ , \ \ \beta = \frac{1}{12} \cdot N_{d-1} - \frac{u+v}{N} \ \ , \tag{40}$$

$$N_{d-1} = \frac{1}{N^2} \cdot \left[1 - (1-N)^N \right] + \frac{1}{2} \cdot (N-2)(N+1) \ \ . \tag{41}$$

(The derivation of these are summarized in the Appendix D). We write the final result of the genus one partition function F_1 of the Calabi-Yau d-fold embedded in the projective space CP^{d+1},

$$\mathsf{F}_1 = \frac{1}{2} \cdot \log \left[e^{v \cdot \mathcal{K}} \cdot \left(\prod_{n=1}^{\left[\frac{d-1}{2}\right]_G} (G_{n\bar{n}})^{-(d-2\,n)} \right) \times \left| \psi^{\alpha} (1-\psi^N)^{\beta} \right|^2 \right] \ \ , \tag{42}$$

with several parameters (37)-(41). Especially when the parameter $\bar{t}$ tends to infinity with the parameter t fixed , the above F_1 (42) turns into a formula,

$$\mathsf{F}_1(\bar{t} \to \infty) = \frac{1}{2} \cdot \log \left[\left(\frac{\psi}{\varpi_0} \right)^v \cdot (1-\psi^N)^{\beta} \cdot \left(\frac{\partial \psi}{\partial t} \right)^u \right] \ \ . \tag{43}$$

In the next section, we explain the meaning of this formula.

5 Interpretation of the Result

In this section, we explain the meaning of the F_1 obtained in the previous section from the point of view of the topological field theory.

5.1 The Asymmetrical Limit

In order to consider the geometrical meaning of the F_1, we consider the limiting case $\bar{t} \to \infty$. The bosonic part S_0 of the action (7) can be written,

$$S_0 \ = \ t \int_{\Sigma} d^2 z \, g_{i\bar{j}} \, \partial_z X^i \partial_{\bar{z}} X^{\bar{j}} + \bar{t} \int_{\Sigma} d^2 z \, g_{i\bar{j}} \, \partial_{\bar{z}} X^i \partial_z X^{\bar{j}} \ \ ,$$

with appropriate constant shifts of the parameters $(t\,,\,\bar{t})$. Obviously the dominant configuration of the boson field in the asymmetrical limit ($\bar{t}\to\infty$ with t fixed) is the holomorphic mapping from Σ to M,

$$\partial_{\bar{z}}X^i = 0 \ .$$

On the other hand, the S_0 is expressed as,

$$S_0 \;=\; \frac{t+\bar{t}}{2}\cdot\int_\Sigma d^2z\,g_{i\bar{\jmath}}\left(\partial_z X^i\partial_{\bar{z}}X^{\bar{\jmath}} + \partial_{\bar{z}}X^i\partial_z X^{\bar{\jmath}}\right) + \frac{t-\bar{t}}{2}\cdot\int_\Sigma d^2z\,g_{i\bar{\jmath}}\left(\partial_z X^i\partial_{\bar{z}}X^{\bar{\jmath}} - \partial_{\bar{z}}X^i\partial_z X^{\bar{\jmath}}\right) \ .$$

The second term is seen to be an integral of the pulled-back Kähler form e of M to Σ and can be rewritten,

$$\begin{aligned}&\int_\Sigma d^2z\,g_{i\bar{\jmath}}\left(\partial_z X^i\partial_{\bar{z}}X^{\bar{\jmath}} - \partial_{\bar{z}}X^i\partial_z X^{\bar{\jmath}}\right) = -\sqrt{-1}\int_\Sigma X^*(e) \ ,\\ &e := \sqrt{-1}\,g_{i\bar{\jmath}}\,dX^i\wedge dX^{\bar{\jmath}} \ .\end{aligned}$$

(In particular, it is independent of the complex structure of the world sheet Σ). With an appropriate normalization, this gives a degree of the map X. Then the path integral of the bosonic part is reduced to the integrals of the holomorphic maps classified by their definite degrees. Now recall the term coupled with the parameter $\bar{t}$ in (7),

$$\bar{t}\int_\Sigma\left\{\widetilde{Q}_R\,,\left[Q_L\,,\sqrt{-1}\,g_{i\bar{\jmath}}\,\psi^i_R\psi^{\bar{\jmath}}_L\right]\right\} \ . \tag{44}$$

In the limit $\bar{t}\to\infty$, this term should be decoupled in order to give non-vanishing contributions to the F_1 in carrying out the path integration. The natural recipe to decouple this term from the theory is to associate the operators $\widetilde{Q}_R$, Q_L to some BRST operator. In that situation, the above term works trivially and does not contribute in the physical situation. When we combine these operators into one operator $Q^{(+)} = \widetilde{Q}_R + Q_L$, the theory turns back to the A(M)-model. We make a remark; The meaning of the holomorphicity varies when one deforms the complex structure of the Riemann surface Σ. Because the integral of the world sheet moduli is performed in the genus one case, we should take into account of the variation of the complex structure of the world sheet (i.e. the topological gravity). In the next subsection, we study a Calabi-Yau A(M)-model coupled to the two-dimensional topological gravity.

5.2 The A-Model Coupled to the Topological Gravity

We consider a topological gravity system. However we take an attention to the complex moduli of the world sheet mainly and omit the diffeomorphism parts and fix the local Weyl scaling eventually. The action S in this system consists of two parts; the topological gravity

part S_G and the A(M)-matter part S_M coupled with the gravity. Firstly we investigate the matter part S_M [1, 26],

$$
\begin{aligned}
S_M &= \int_\Sigma d^2z \Big[g_{i\bar{j}}\,\partial_{\bar{z}}X^i \partial_z X^{\bar{j}} + \sqrt{-1}\rho_{zi}\left(D_{\bar{z}}\chi^i + \chi_{\bar{z}}^z \partial_z X^i\right) \\
&\quad + \sqrt{-1}\rho_{\bar{z}\bar{\imath}}\,(D_z\chi^{\bar{\imath}} + \chi_z^{\bar{z}}\partial_{\bar{z}}X^{\bar{\imath}}) - \chi^k\chi^{\bar{l}} R^i_{jk\bar{l}}\rho_{zi}\rho_{\bar{z}}^j + \chi_{\bar{z}}^z\chi_z^{\bar{z}}\rho_{zi}\rho_{\bar{z}}^i \Big] \ , \qquad (45)
\end{aligned}
$$

where the fields $\chi_{\bar{z}}^z$, $\chi_z^{\bar{z}}$ are superpartners of the complex structure J^μ_ν on the Σ and its conjugate. The BRST transformations of these fields are collected as,

$$
\begin{aligned}
&\delta X^i = \chi^i \ \ , \ \ \delta\chi^i = 0 \ \ , \\
&\delta X^{\bar{\imath}} = \chi^{\bar{\imath}} \ \ , \ \ \delta\chi^{\bar{\imath}} = 0 \ \ , \\
&\delta\rho_z^{\bar{\imath}} = \sqrt{-1}\partial_z X^{\bar{\imath}} - \Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}\chi^{\bar{j}}\rho_z^{\bar{k}} \ \ , \\
&\delta\rho_{\bar{z}}^i = \sqrt{-1}\partial_{\bar{z}} X^i - \Gamma^i_{jk}\chi^j\rho_{\bar{z}}^k \ \ , \\
&\delta J_{\bar{z}}^z = -2\sqrt{-1}\chi_{\bar{z}}^z \ \ , \ \ \delta\chi_{\bar{z}}^z = 0 \ \ , \\
&\delta J_z^{\bar{z}} = 2\sqrt{-1}\chi_z^{\bar{z}} \ \ , \ \ \delta\chi_z^{\bar{z}} = 0 \ \ .
\end{aligned}
$$

This system is quasi-topological and the weak coupling limit gives exact results. In fact, the path integral is dominated by the holomorphic instanton configurations, which are defined in the space $\mathcal{C}_\Sigma \times \mathrm{Map}(\Sigma\,,\,\mathrm{M})$ as,

$$
Inst := \{(J\,,\,f)\,;\ f:\Sigma_J \to \mathrm{M} \quad (\text{holomorphic})\} \ \ .
$$

The complex structure $\mathcal{C}_\Sigma$ of the Riemann surface Σ is described by the $(1,1)$ tensor J on Σ. (We use the notation in [26]). The holomorphic property of the map f is written as,

$$
\frac{1}{2}dx^\mu(\delta_\mu^\nu + \sqrt{-1}J_\mu^\nu)\partial_\nu f^i = 0 \ \ ,
$$

where the $\{x^\mu\}$ is the local real coordinate on Σ.

Now let us consider an infinitesimal deformation of the configuration $(J\,,\,f) \to (J+\delta J\,,\,f+\delta f)$. The deformed configuration remains in the space $Inst$ if the following equation is satisfied,

$$
\partial_{\bar{z}}(\delta f^i) + \frac{\sqrt{-1}}{2}\delta J_{\bar{z}}^z \partial_z f^i = 0 \ \ .
$$

The tangential direction of the $Inst$ is controlled by the ghost zero modes $(\chi^i\,,\,\chi_{\bar{z}}^z)$ with the relation,

$$
\partial_{\bar{z}}\chi^i + \chi_{\bar{z}}^z\partial_z f^i = 0 \ \ . \qquad (46)
$$

The moduli space of the genus g Riemann surface with s-distinct punctures is described by the fermionic fields $\chi_{\bar{z}}^z$ and $\chi_z^{\bar{z}}$. On the other hand, the transversal direction to the $Inst$ is

represented by the pair $(v^i\,,\,\omega)$. The v^i is the variation of the map f^i ($\delta f^i = v^i$) and belongs to the space $(H^0(f^*T\mathrm{M}))^\perp$. The ω is a Beltrami differential with properties,

$$\delta J^z_{\bar z} = -2\sqrt{-1}\,\omega^z_{\bar z}\ ,$$
$$\omega^z_{\bar z}\partial_z f^i \neq 0\ .$$

Also their complex conjugate $(\bar v\,,\,\bar\omega)$ satisfies a condition,

$$\bar v \in (H^0(f^*T^*M))^\perp\ ,$$

and the $\bar\omega$ can be expressed by a holomorphic map f^i and an anti-ghost ρ_{zi} as,

$$g_{\bar z z}\bar\omega^{\bar z}_z = \rho_{zi}\partial_z f^i\ .$$

When one deforms the pair $(J\,,\,f)$ infinitesimally into $(J+\delta J\,,\,f+\delta f)$, the action also changes. The quadratic parts with respect to the variation fields can be obtained,

$$\begin{aligned}
S^{(2)} &= S_1 + S_2\ ,\\
S_1 &:= \int_\Sigma d^2z\Big\{\sqrt{-1}\rho_{zi}\left(\partial_{\bar z}\chi^i + \chi^z_{\bar z}\partial_z f^i\right) + \sqrt{-1}\rho_{\bar z\bar\imath}\left(\partial_z\chi^{\bar\imath} + \chi^{\bar z}_z\partial_{\bar z}f^{\bar\imath}\right)\\
&\quad -R^i_{j\,k\bar l}\rho_{zi}\rho^j_{\bar z}\chi^k\chi^{\bar l} + \rho_{zi}\rho^i_{\bar z}\chi^z_{\bar z}\chi^{\bar z}_z\Big\}\ ,\\
S_2 &= \int_\Sigma d^2z\Big\{-g^{\bar\imath j}v_{\bar\imath}D_{\bar z}D_z\bar v_j + D_{\bar z}v_{\bar\imath}\cdot\bar\omega^{\bar z}_z\partial_{\bar z}f^{\bar\imath} + \omega^z_{\bar z}\partial_z f^i\cdot D_z\bar v_i\\
&\quad + g_{i\bar\jmath}\,\omega^z_{\bar z}\partial_z f^i\cdot\bar\omega^{\bar z}_z\partial_{\bar z}f^{\bar\jmath} - \sqrt{-1}v^i\tilde\varphi_{\bar z zi} + \sqrt{-1}\overline{\tilde\varphi}_{z\bar z\bar\imath}\bar v^{\bar\imath}\\
&\quad -\sqrt{-1}\omega^z_{\bar z}D_z\chi^i\cdot\rho_{zi} + \sqrt{-1}\rho_{\bar z\bar\imath}D_{\bar z}\chi^{\bar\imath}\cdot\bar\omega^{\bar z}_z\Big\}\ ,\\
&\quad \tilde\varphi_{\bar z zi} := \chi^k\partial_{\bar z}f^{\bar l}\cdot R^j_{i\,k\bar l}\rho_{zj} - D_z(\chi^z_{\bar z}\rho_z)_i\ ,\\
&\quad \overline{\tilde\varphi}_{z\bar z\bar\imath} := \rho_{\bar z\bar\jmath}\chi^{\bar k}\partial_z f^l\cdot R^{\bar\jmath}_{\bar\imath\,\bar k l} - D_{\bar z}(\rho_{\bar z}\chi^{\bar z}_z)_{\bar\imath}\ .
\end{aligned}$$

The ghosts are decomposed into sums of the tangential parts and the transversal parts of the *Inst*. When one carries out the path integrals in the transversal parts, the contributions from the bosonic parts and the fermionic parts cancel with each other. Then one obtains an effective action of the zero mode components,

$$\begin{aligned}
S_{eff} &= \int_\Sigma\Big\{-\chi^k\chi^{\bar l}R^i_{j\,k\bar l}\rho_{zi}\rho^j_{\bar z} + \chi^z_{\bar z}\chi^{\bar z}_z\rho_{zi}\rho^i_{\bar z} + \overline{\tilde\varphi}_{z\bar z}{}^i\{G(\tilde\varphi)\}_i\\
&\quad -\left(\bar\rho(\bar D\bar\chi) - \overline{\tilde\varphi}G\bar D\bar f_\sharp\right)\left({}^t f_\sharp(1-DG\bar D)\bar f_\sharp\right)^{-1}\left((D\chi)\rho + {}^t f_\sharp DG(\tilde\varphi)\right)\Big\}\ ,
\end{aligned}\tag{47}$$

where the fermions $(\chi^i\,,\,\rho^i_{\bar z}\,,\,\rho_{zi})$ satisfy the zero mode conditions,

$$\partial_{\bar z}\chi^i + \chi^z_{\bar z}\partial_z f^i = 0\ ,$$
$$\partial_{\bar z}\rho_{zi} = 0\ ,\quad \rho_{zi}\partial_z f^i = 0\ .$$

Also the symbol "G" means a Green's function of the operator $\bar{D}D$ and the mappings $f_\sharp$ and $\bar{f}_\sharp$ are defined on the ω and $\bar{\omega}$ respectively,

$$\begin{aligned}(f_\sharp \omega)_{\bar{z}\bar{\imath}} &:= g_{i\bar{\jmath}} \partial_z f^j \cdot \omega_{\bar{z}}^z \ , \\ (\bar{f}_\sharp \bar{\omega})_{zi} &:= g_{i\bar{\jmath}} \partial_{\bar{z}} f^{\bar{\jmath}} \cdot \bar{\omega}_z^{\bar{z}} \ .\end{aligned}$$

In the next subsection, we give a geometrical meaning of the effective action (47) for the (anti)-ghost zero modes.

5.3 The Geometrical Meaning

In the last subsection, we obtain an effective action for the (anti)-ghost zero modes. If one integrates this with respect to the (anti)-ghost zero modes, the result gives a top Chern class of the vector bundle $\mathcal{V}$ over the holomorphic instanton space $Inst$. The fiber of the bundle $\mathcal{V}$ is spanned by the anti-ghost zero modes,

$$\partial_{\bar{z}} \rho_{zi} = 0 \ , \ \ \rho_{zi} \partial_z f^i = 0 \ .$$

On the bundle $\mathcal{V}$, a covariant derivative ∇ is defined as [26],

$$\begin{aligned}\nabla \rho \ = \ & \Big\{ \delta \rho_i - \delta f^k \Gamma_{ki}^j \rho_j + \frac{\sqrt{-1}}{2} \cdot d\bar{z} \, \delta J_{\bar{z}}^z \rho_{zi} \\ & - dz \, D_z (G(\tilde{\varphi}))_i + dz \, (df)^{-1} \Big\{ (D\delta^{1,0} f) \rho + {}^t f_\sharp D G(\tilde{\varphi}) \Big\} \Big\} \otimes dX^i \Big|_{(J,f)} \ ,\end{aligned}$$

where $(df)^{-1}$ is a Green's function of the operator df,

$$df \ ; \ \ \xi_{zi} \mapsto \xi_{zi} \partial_z f^i \ .$$

The second and the third terms originate from the variations of the map $f + \delta f$ and the complex structure $J + \delta J$ respectively. The presence of the fourth and the fifth terms guarantees the holomorphicity and the conormal property of the ρ with respect to the $(J + \delta J \, , \, f + \delta f)$. By using this connection on $\mathcal{V}$, the effective action can be rewritten as,

$$S_{eff} = - \Big(\rho \, , \, \Big[\nabla^{0,1} \, , \nabla^{1,0} \Big] \rho \Big) \ ,$$

with $\delta f^i = \chi^i$, $\delta J_{\bar{z}}^z = -2\sqrt{-1} \chi_{\bar{z}}^z$. The inner product is defined by the hermitian metrics $g_{i\bar{\jmath}}$ and $g_{z\bar{z}}$. That is to say, the effective action is a bilinear form of the anti-ghost zero modes with the curvature of the vector bundle $\mathcal{V}$. Thus the integral of the anti-ghost zero modes gives a top Chern class $c_T(\mathcal{V})$ of the bundle $\mathcal{V}$. The remaining ghost zero modes $(\chi^i \, , \, \chi_{\bar{z}}^z)$ with the relation (46) govern the tangent space of the holomorphic instanton space. In other

words, they describe the tangent space of the moduli space of stable maps $\overline{\mathcal{M}}_{g,s}(\mathrm{M}\,,\,n)$. This moduli space is defined by a set of the s-distinct marked points $(x_1, x_2, \cdots, x_s)$ on Σ and a holomorphic map f from the Riemann surface Σ to the target space M. The "stable" means that a map $f\,;\,\Sigma \rightarrow \mathrm{M}$ has no non-trivial first order infinitesimal automorphisms, identical on M. In other words, it means that the automorphism group of $(\Sigma\,;\,x_1\,\cdots, x_s\,;\,f)$ is finite. (Concretely the condition means that every component of Σ of genus 0 (resp. 1) which maps to a point must have at least 3 (resp. 1) marked or singular points). The moduli space of stable maps to M of curves is defined as,

$$\overline{\mathcal{M}}_{g,s}(\mathrm{M}\,,\,n) := \{(\Sigma\,;\,x_1, \cdots, x_s\,;\,f)\}\,/ \cong \;, \tag{48}$$

where "$\cong$" is the action of the finite automorphism group of $(\Sigma\,;\,x_1, \cdots, x_s\,;\,f)$. Also the degree n of the map f is a non-negative integer determined by the homology class $H_2(\mathrm{M}\,;\,\mathbf{Z})$ and the homotopy class of the map.

Next we consider the topological gravity part S_G. For each marked point x_i on Σ, a holomorphic cotangent space $T^*_{x_i}\Sigma$ is defined. When one deforms the complex structure of Σ, the meaning "holomorphicity" varies. Then a complex line bundle $\mathcal{L}_{(i)}$ over the base space $\overline{\mathcal{M}}_{g,s}(\mathrm{M}\,,\,n)$ with the fiber $T^*_{x_i}\Sigma$ is introduced naturally. When one considers a correlator $\langle \cdots \sigma_{n_i}(\omega_i) \cdots \rangle$ containing a n_i-th gravitational descendant $\sigma_{n_i}(\omega_i)$ of a cohomology element $\omega_i \in H^{2q_i}(\mathrm{M})$, it contributes after the path integral as,

$$c_1(\mathcal{L}_{(i)})^{n_i}\, f^*(\omega_i)(x_i) \;\;.$$

Collecting the matter parts and the gravity parts together, we can write the correlation functions in terms of geometry,

$$\langle \sigma_{n_1}(\omega_1)\sigma_{n_2}(\omega_2) \cdots \sigma_{n_s}(\omega_s) \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}(\mathrm{M},n)} c_T\,(\mathcal{V}_{g,s,n}) \prod_{i=1}^{s} c_1(\mathcal{L}_{(i)})^{n_i}\, f^*(\omega_i)(x_i) \;\;,$$

provided that the following selection rule is satisfied,

$$\sum_{i=1}^{s}(n_i + q_i) = (\dim \mathrm{M} - 3)(1-g) + \int_\Sigma f^* c_1(\mathrm{M}) + s \;\;.$$

For our genus one case of the Calabi-Yau $A(\mathrm{M})$-model, the first and the second terms vanish and then the selection rule depends on the number of the marked points s only.

Now we express the $\partial_t \mathsf{F}_1(\bar{t} \rightarrow \infty)$ in geometrical terms as,

$$\begin{aligned} \partial_t \mathsf{F}_1 &= \sum_{n=0}^{\infty} q^n \langle \sigma_0(\omega) \rangle_{1,n} \qquad\quad (g=1\;,\;\omega \in H^{1,1}(\mathrm{M})) \\ &= \sum_{n=0}^{\infty} q^n \left\{ \int_{\overline{\mathcal{M}}_{1,1}(\mathrm{M},n)} c_T(\mathcal{V}_{1,1,n})\, f^*(\omega) \right\} \;\;. \end{aligned} \tag{49}$$

Particularly the degree 0 part associated to the Kähler form ω can be calculated,

$$\langle \sigma_0(\omega) \rangle_{1,0} = -\frac{1}{24}\int_{\mathrm{M}} \omega \wedge c_{d-1}(\mathrm{M}) \ .$$

(This can be used to fix the normalization of the F_1).

Finally we make several remarks; the holomorphic maps for $g \leq 1$ are isolated only at $\dim \mathrm{M} = 3$. In this case, one can count the number of the curves embedded in M unambiguously. But the $\dim \mathrm{M} > 3$ cases, the anti-ghost zero modes exist and the maps appear as families. In such cases, the counting the number of the "individual" curves no longer lose its meaning and one should interpret $\partial_t \mathsf{F}_1(\bar{t} \to \infty)$ as an integral (49) of the top Chern class over the moduli space $\overline{\mathcal{M}}_{1,1}(\mathrm{M}, n)$. Also for $\dim \mathrm{M} = 3$ cases, the $\partial_t \mathsf{F}_1(\bar{t} \to \infty)$ have contributions from the rational maps as well as elliptic maps because of the bubbling of the torus [14]. In our higher dimensional cases ($\dim M > 3$), the similar bubbling phenomena are expected to appear. However the explanation of these possible bubbling needs some new formulation in moduli spaces of holomorphic instanton families which may have singular or non-smooth configurations. It remains an open problem.

6 Conclusion

In this article, we have investigated some properties of the Calabi-Yau d-fold M embedded in CP^{d+1} subject to the assumption of the existence of the mirror symmetries. We introduce a new quasi-topological field theory $A^*(\mathrm{M})$-model associated to the M. This model is compared to the $A(\mathrm{M})$-model. By the analysis in the AA^*-fusion of these two models, two point functions of the moduli space associated to the non-linear sigma model are investigated. A set of the two point correlators satisfies a set of equations (AA^*-equation) for the Kähler manifold M. Because we switch of all perturbation operators on the topological theory except for marginal ones $\mathcal{O}^{(1)}$, $\bar{\mathcal{O}}^{(\bar{1})}$ associated with a Kähler form of M, the AA^*-equation is characterized by three point functions $\mathcal{K}_{l-1} := \left\langle \mathcal{O}^{(d-l)} \middle| \mathcal{O}^{(1)} \middle| \mathcal{O}^{(l-1)} \right\rangle$ and their conjugates $\bar{\mathcal{K}}_{l-1} := \left\langle \bar{\mathcal{O}}^{(\overline{d-l})} \middle| \bar{\mathcal{O}}^{(\bar{1})} \middle| \bar{\mathcal{O}}^{(\overline{l-1})} \right\rangle$. (The author calculated these three point functions on the sphere in the previous paper [13]). That is to say, the fusion structure constants in the $A(\mathrm{M})$-model control the system,

$$\begin{aligned} \mathcal{O}^{(1)} \cdot \mathcal{O}^{(l-1)} &= \mathcal{K}_{l-1}\, \mathcal{O}^{(l)} \ , \\ \mathcal{O}^{(1)} \cdot \mathcal{O}^{(d)} &= 0 \ . \end{aligned}$$

For our Calabi-Yau case, this AA^*-equation turns out to be a non-affine A-type Toda equation. This non-affine property stems from the nilpotency of the operator products,

$\mathcal{O}^{(1)} \cdot \mathcal{O}^{(d)} = 0$. This nilpotency originates in the vanishing first Chern class of the Calabi-Yau manifold. For the CP^{d+1} case, its first Chern class does not vanish and then the AA^*-equation is an affine Toda equation system. In order to obtain genus one partition function of the non-linear sigma model described by the Lagrangian $L(t\,,\,\bar{t})$, we used the data of the two correlators. By taking an asymmetrical limit $\bar{t} \to \infty$ and t is fixed, the A^*-model part is decoupled and the above partition function is reduced to that of the A(M)-matter coupled with the the topological gravity at the stringy one loop level. The coefficients of the $\mathsf{F}_1(\bar{t} \to \infty)$ with respect to the indeterminate $q := e^{2\pi i t}$ represents integrals of the top Chern class of the vector bundle $\mathcal{V}$ over the moduli space of the stable curves with definite degrees n. In this paper we used the mirror conjecture and our results should be verified by the mathematical methods in enumerative geometry [27].

Appendix A

The AA^*-Fusion

Firstly let us introduce a set of connections $\mathcal{A}, \overline{\mathcal{A}}$ defined as,

$$
\begin{aligned}
\mathcal{A}_{j\bar{k}} = \left\langle \bar{\mathcal{O}}^{(\bar{k})} \middle| \mathcal{A} \middle| \mathcal{O}^{(j)} \right\rangle := \left\langle \bar{\mathcal{O}}^{(\bar{k})} \middle| \partial_t \middle| \mathcal{O}^{(j)} \right\rangle \ , \\
\overline{\mathcal{A}}_{j\bar{k}} = \left\langle \bar{\mathcal{O}}^{(\bar{k})} \middle| \overline{\mathcal{A}} \middle| \mathcal{O}^{(j)} \right\rangle := \left\langle \bar{\mathcal{O}}^{(\bar{k})} \middle| \partial_{\bar{t}} \middle| \mathcal{O}^{(j)} \right\rangle \ , \\
\partial_t := \frac{\partial}{\partial t} \ , \ \ \partial_{\bar{t}} := \frac{\partial}{\partial \bar{t}} \ ,
\end{aligned}
$$

where $\mathcal{O}^{(j)}$, $\bar{\mathcal{O}}^{(\bar{k})}$ are observables of the A-model , A^*-model respectively. Using these connections, we define covariant derivatives,

$$
\begin{aligned}
D_t &:= \partial_t - \mathcal{A} \ , \\
\bar{D}_{\bar{t}} &:= \partial_{\bar{t}} - \overline{\mathcal{A}} \ .
\end{aligned}
$$

From the path integral representation, the next relation follows,

$$
\begin{aligned}
\overline{\mathcal{A}}_{jk} &= \left\langle \mathcal{O}^{(k)} \middle| \overline{\mathcal{A}} \middle| \mathcal{O}^{(j)} \right\rangle \\
&= \left\langle \mathcal{O}^{(k)} \middle| \partial_{\bar{t}} \middle| \mathcal{O}^{(j)} \right\rangle = 0 \ .
\end{aligned} \tag{50}
$$

Also we define a notation,

$$
\mathcal{A}_{jk} := \left\langle \mathcal{O}^{(k)} \middle| \partial_t \middle| \mathcal{O}^{(j)} \right\rangle \ ,
$$

and consider a derivative of this $\partial_{\bar{t}} \mathcal{A}_{jk}$ with respect to the parameter $\bar{t}$. Then we obtain,

$$
\partial_{\bar{t}} \mathcal{A}_{jk} = \partial_{\bar{t}} \mathcal{A}_{jk} - \partial_t \overline{\mathcal{A}}_{jk}
$$

$$
\begin{aligned}
&= \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \left\{ Q_R \,, \left[\widetilde{Q}_L \,, \bar{\mathcal{O}}^{(\bar{1})} \right] \right\} \right) \Big| \left(\int_{\Sigma_R} \left\{ \widetilde{Q}_R \,, \left[Q_L \,, \mathcal{O}^{(1)} \right] \right\} \right) \mathcal{O}^{(j)} \right\rangle \\
&\quad - \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \left\{ Q_R \,, \left[\widetilde{Q}_L \,, \mathcal{O}^{(1)} \right] \right\} \right) \Big| \left(\int_{\Sigma_R} \left\{ \widetilde{Q}_R \,, \left[Q_L \,, \bar{\mathcal{O}}^{(\bar{1})} \right] \right\} \right) \mathcal{O}^{(j)} \right\rangle \\
&= \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \bar{\mathcal{O}}^{(\bar{1})} \right) \Big| \left(\int_{\Sigma_R} \partial_z \bar{\partial}_{\bar{z}} \mathcal{O}^{(1)} \right) \mathcal{O}^{(j)} \right\rangle \\
&\quad - \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \mathcal{O}^{(1)} \right) \Big| \left(\int_{\Sigma_R} \partial_z \bar{\partial}_{\bar{z}} \bar{\mathcal{O}}^{(\bar{1})} \right) \mathcal{O}^{(j)} \right\rangle \quad ,
\end{aligned} \tag{51}
$$

by using the relations,

$$
\begin{aligned}
\left\{ Q_L \,, \widetilde{Q}_L \right\} &= \partial_z \quad , \\
\left\{ Q_R \,, \widetilde{Q}_R \right\} &= \bar{\partial}_{\bar{z}} \quad .
\end{aligned}
$$

The first term in (51) is rewritten as,

$$
\begin{aligned}
&\left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \bar{\mathcal{O}}^{(\bar{1})} \right) \Big| \left(\int_{\Sigma_R} \partial_z \bar{\partial}_{\bar{z}} \mathcal{O}^{(1)} \right) \mathcal{O}^{(j)} \right\rangle \\
&= - \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \bar{\mathcal{O}}^{(\bar{1})} \right) \Big| \left(\oint_C \partial_n \mathcal{O}^{(1)} \right) \mathcal{O}^{(j)} \right\rangle \\
&= - \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \bar{\mathcal{O}}^{(\bar{1})} \right) \Big| \left(H \oint_C \mathcal{O}^{(1)} \right) \Big| \mathcal{O}^{(j)} \right\rangle \quad ,
\end{aligned}
$$

where C is the boundary of Σ_R and ∂_n is the normal derivative along the cylindrical direction of Σ and H is a hamiltonian along this direction of Σ,

$$
\partial_n \mathcal{O}^{(i)} = \left[H \,, \mathcal{O}^{(i)} \right] \quad .
$$

Furthermore the above formula is re-expressed as,

$$
\begin{aligned}
&- \left\langle \mathcal{O}^{(k)} \left(\int_{\Sigma_L} \bar{\mathcal{O}}^{(\bar{1})} \right) \Big| \left(H \oint_C \mathcal{O}^{(1)} \right) \Big| \mathcal{O}^{(j)} \right\rangle \\
&= - \int_0^T d\tau \left\langle \mathcal{O}^{(k)} \Big| \left(\oint_{C'} \bar{\mathcal{O}}^{(\bar{1})} \right) H \, e^{-\tau H} \left(\oint_C \mathcal{O}^{(1)} \right) \Big| \mathcal{O}^{(j)} \right\rangle \quad ,
\end{aligned}
$$

where the integration $\int d\tau$ is over the length of the left cylinder Σ_L. Finally taking a long-cylindrical limit $T \to \infty$, the $\partial_{\bar{t}} \mathcal{A}_{jk}$ becomes,

$$
\begin{aligned}
&\left\langle \mathcal{O}^{(k)} \Big| \left(\oint_{C'} \bar{\mathcal{O}}^{(\bar{1})} \right) e^{-TH} \left(\oint_C \mathcal{O}^{(1)} \right) \Big| \mathcal{O}^{(j)} \right\rangle_{(T \to \infty)} \\
&- \left\langle \mathcal{O}^{(k)} \Big| \left(\oint_{C'} \mathcal{O}^{(1)} \right) e^{-TH} \left(\oint_C \bar{\mathcal{O}}^{(\bar{1})} \right) \Big| \mathcal{O}^{(j)} \right\rangle_{(T \to \infty)} \\
&= \beta^2 \left(\bar{C}_{\bar{t}} \right)^n_{\ j} (C_t)_{nk} - \beta^2 (C_t)^n_{\ j} \left(\bar{C}_{\bar{t}} \right)_{nk} \quad ,
\end{aligned}
$$

where C_t, $\bar{C}_{\bar{t}}$ are three point couplings of the A-model, the A^*-model respectively and are defined as,

$$
\begin{aligned}
(C_t)_{lm} &:= \left\langle \mathcal{O}^{(m)} \Big| \mathcal{O}^{(1)} \Big| \mathcal{O}^{(l)} \right\rangle \quad , \\
\left(\bar{C}_{\bar{t}} \right)_{\bar{l}\bar{m}} &:= \left\langle \bar{\mathcal{O}}^{(\bar{m})} \Big| \bar{\mathcal{O}}^{(\bar{1})} \Big| \bar{\mathcal{O}}^{(\bar{l})} \right\rangle \quad .
\end{aligned}
$$

Also the symbol $\left(\bar{C}_{\bar{t}}\right)_j^{\ k}$ means that,

$$\left(\bar{C}_{\bar{t}}\right)_j^{\ k} = \mathsf{g}_{j\bar{n}} \left(\bar{C}_{\bar{t}}\right)_{\bar{m}}^{\ \bar{n}} \mathsf{g}^{\bar{m}k} \ \ .$$

For the hermitian metric $\mathsf{g}_{l\bar{m}}$ and the covariant derivative $D_t = \partial_t - \mathcal{A}$, the next relation is understood,

$$D_t \mathsf{g}_{l\bar{m}} = 0 \ \ .$$

That is to say, the connection can be written as,

$$\mathcal{A}_j^l = -\mathsf{g}_{j\bar{n}} (\partial_t \mathsf{g}^{-1})^{\bar{n}l} \ \ .$$

Now we raise a holomorphic index of the $\mathcal{A}_{jk}$ by multiplying the topological metric η^{lk} and obtain a final result,

$$\begin{aligned} \partial_{\bar{t}} \mathcal{A}_j^l &= -\partial_{\bar{t}} \Big(\mathsf{g}\, \partial_t \mathsf{g}^{-1} \Big)_j^{\ l} \\ &= -\beta^2 \Big[C_t \,, \mathsf{g}\, \bar{C}_{\bar{t}}\, \mathsf{g}^{-1} \Big]_j^{\ l} \ \ . \end{aligned}$$

Appendix B

The Three Point Functions on the Sphere

In this appendix B, we derive the three point functions on the sphere of the d-fold (2) under the mirror symmetry. (More detail can be seen in [13]).

Firstly we take notice of the Hodge structure of the primary horizontal subspace $H^d(\mathrm{W})$ of the mirror manifold W paired with M. The deformation of the complex structure of W is controlled by the variation of the Hodge decomposition of $H^d(W)$ and its information is given by the period matrix P of W. This period matrix is defined by using homology d-cycles $\gamma_j \in H_d(W)$, cohomology elements,

$$\begin{aligned} &\alpha_i := \Theta_z^i \Omega \in \mathcal{F}^{d-i} = H^{d,0} \oplus H^{d-1,1} \oplus \cdots \oplus H^{d-i,i} \ \ , \\ &N := d+2 \ , \ z := (N\psi)^{-N} \ , \ \Theta_z := z \cdot \frac{d}{d\,z} \ \ . \end{aligned}$$

Its matrix elements P_{ij} $(0 \le i \le d \,,\, 0 \le j \le d)$ are expressed as,

$$P_{ij} := \int_{\gamma_j} \alpha_i \ \ .$$

The $\alpha_0 = \Omega$ is a globally defined nowhere-vanishing holomorphic d-form of W and can be expressed for the Fermat type hypersurface p as,

$$\begin{aligned} &\Omega := \int_\gamma \frac{d\mu}{\mathsf{p}} \ \ , \\ &d\mu := \sum_{a=1}^{d+2} (-1)^{a-1} X_a dX_1 \wedge dX_2 \wedge \cdots \wedge \widehat{dX_a} \wedge \cdots \wedge dX_{d+2} \ \ , \end{aligned}$$

where γ is a small one-dimensional cycle winding around the hypersurface defined as a zero locus of p. Using this explicit formula of the Ω, we obtain a differential equation satisfied by P,

$$
\begin{aligned}
&\Theta_z \, P(z) = \widetilde{C}_z \, P(z) \ , \\
&\widetilde{C}_z := \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ \sigma_{N-1} & \sigma_{N-2} & \sigma_{N-3} & \cdots & \sigma_2 & \sigma_1 \end{pmatrix} , \\
&\sigma_m := \frac{N^N z}{1-N^N z} \sum_{1 \le n_1 < n_2 < \cdots < n_m \le N-1} \frac{n_1}{N} \cdot \frac{n_2}{N} \cdot \cdots \cdot \frac{n_m}{N} \ .
\end{aligned}
$$

Each component P_{ij} of P is obtained,

$$
\begin{aligned}
&P_{0j} = \varpi_j(z) := \sum_{l=0}^{j} \frac{1}{l!} \left(\frac{\log z}{2\pi i} \right)^l \times \sum_{m=0}^{\infty} b_{j-l,m} \cdot z^m \ , \\
&b_{n,m} := \frac{1}{n!} \left(\frac{1}{2\pi i} \cdot \frac{\partial}{\partial \rho} \right)^n \left\{ \frac{\Gamma(N(m+\rho)+1)}{\Gamma(N\rho+1)} \cdot \left[\frac{\Gamma(\rho+1)}{\Gamma(m+\rho+1)} \right]^N \right\} \Bigg|_{\rho=0} \ , \\
&P_{ij} = \int_{\gamma_j} \Theta_z^i \Omega = \Theta_z^i \varpi_j \ .
\end{aligned}
$$

In the equation, the derivative Θ_z induces the variation of the complex structure of W and the structure of the Hodge decomposition is deformed. Also each entry of the l-th row of the P belongs to the $\mathcal{F}^{d-l}$. So the equation describes a linear representation of the infinitesimal deformation of the complex structure of W.

Next by introducing a new variable $t = \omega_1(z)$, we rewrite the above differential equation on the upper triangular matrix Φ,

$$
\begin{aligned}
&\partial_t \Phi(t) = C_t \, \Phi(t) \ , \quad \Phi_{lm} := \int_{\gamma_m} \tilde{\alpha}_l(t) \ , \\
&C_t := \begin{pmatrix} 0 & \mathcal{K}_0 & & & & & O \\ & 0 & \mathcal{K}_1 & & & & \\ & & 0 & \mathcal{K}_2 & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & \mathcal{K}_{d-2} & \\ & & & & & 0 & \mathcal{K}_{d-1} \\ O & & & & & & 0 \end{pmatrix} , \\
&\tilde{\alpha}_0(t) := \frac{1}{\varpi_0} \alpha_0 \ ,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\alpha}_l(t) := \frac{1}{\mathcal{K}_{l-1}}\partial_t\frac{1}{\mathcal{K}_{l-2}}\partial_t\cdots\partial_t\frac{1}{\mathcal{K}_1}\partial_t\frac{1}{\mathcal{K}_0}\partial_t\left(\frac{\alpha_0}{\varpi_0}\right) \ , \ (1 \le l \le d) \ , \\
&\mathcal{K}_0 := \partial_t\,\omega_1 = 1 \ , \ (t \equiv \omega_1) \\
&\mathcal{K}_m := \partial_t\frac{1}{\mathcal{K}_{m-1}}\partial_t\frac{1}{\mathcal{K}_{m-2}}\partial_t\cdots\partial_t\frac{1}{\mathcal{K}_1}\partial_t\frac{1}{\mathcal{K}_0}\partial_t\,\omega_{m+1} \ , \ (1 \le m \le d-1) \ .
\end{aligned}
$$

From this equation, we can read the action of the differential operator ∂_t on the cohomology basis $\tilde{\alpha}_l(t)$,

$$
\begin{aligned}
&\partial_t\tilde{\alpha}_{j-1}(t) = \mathcal{K}_{j-1}(t)\,\tilde{\alpha}_j(t) \ , \ (1 \le j \le d) \ , \\
&\partial_t\tilde{\alpha}_d(t) = 0 \ , \\
&\mathcal{K}_{j-1}(t) = \partial_t\Phi_{j-1\,,\,j} \ , \ (1 \le j \le d) \ .
\end{aligned}
$$

Because the $t(z)$ is a mirror map and couples with a Kähler form $J := t \cdot e$ of M, the derivative ∂_t can be interpreted as an insertion of the operator $\mathcal{O}^{(1)}$ associated to the Kähler form e in the A(M)-model terms. We translate these relations into the operator structures of the A(M)-model,

$$
\begin{aligned}
&\mathcal{O}^{(1)}\mathcal{O}^{(j-1)} = \mathcal{K}_{j-1}(t)\mathcal{O}^{(j)} \ , \ (1 \le j \le d) \ , \\
&\mathcal{O}^{(1)}\mathcal{O}^{(d)} = 0 \ ,
\end{aligned}
$$

for A(M)-model operators $\mathcal{O}^{(i)} \in H^{i,i}_J(M)\,,\,(1 \le i \le d)$. The above operator product structure of the A(M)-model observables is meaningful when one defines correlation functions in the following way,

$$
\begin{aligned}
\left\langle \mathcal{O}^{(1)}\mathcal{O}^{(j-1)}\cdots\right\rangle &:= \int\mathcal{D}[X,\chi,\rho]\,\mathcal{O}^{(1)}\mathcal{O}^{(j-1)}\cdots e^{-L_A} \ , \\
L_A &:= -\sqrt{-1}\int_\Sigma X^*(e) + \int_\Sigma d^2z\,\left\{Q^{(+)}\,,V^{(-)}\right\} \ ,
\end{aligned}
$$

where $Q^{(+)}$ is a BRST charge of the A(M)-model and $V^{(-)}$ is given in (4). Thus we can obtain the fusion couplings $\{\mathcal{K}_l\} := \left\langle\mathcal{O}^{(1)}\mathcal{O}^{(l)}\mathcal{O}^{(d-l-1)}\right\rangle$,

$$
\begin{aligned}
&\mathcal{K}_0 = 1 \ , \\
&\mathcal{K}_l = \partial_t\frac{1}{\mathcal{K}_{l-1}}\partial_t\frac{1}{\mathcal{K}_{l-2}}\partial_t\cdots\partial_t\frac{1}{\mathcal{K}_2}\partial_t\frac{1}{\mathcal{K}_1}\partial_t\frac{1}{\mathcal{K}_0}\partial_t\,S_{l+1}(\tilde{x}_1\,,\,\tilde{x}_2\,,\,\cdots\,,\,\tilde{x}_{l+1}) \ , \ (1 \le l \le d-1) \ , \\
&\tilde{x}_n := \frac{1}{n!}D^n_\rho\log\hat{\varpi}_0(z\,;\,\rho)\Big|_{\rho=0} \ , \ \mathcal{D}_\rho := \frac{1}{2\pi\,i}\cdot\frac{\partial}{\partial\rho} \ , \\
&\hat{\varpi}_0(z,\rho) := \sum_{m=0}^{\infty}\frac{\Gamma(N(m+\rho)+1)}{\Gamma(N\rho+1)}\cdot\left[\frac{\Gamma(\rho+1)}{\Gamma(m+\rho+1)}\right]^N\cdot z^{m+\rho} \ ,
\end{aligned}
$$

where the function "S_n" is the Schur function defined as the coefficients in the following expansion,

$$
\sum_{n=0}^{\infty}S_n(x_1,x_2,\cdots,x_n)u^n := \exp\left(\sum_{m=1}^{\infty}x_m u^m\right) \ .
$$

We write down expressions of these couplings $\mathcal{K}_l$ in a series with respect to a parameter $q := e^{2\pi i\, t}$, $t = S_1(\tilde{x}_1) = \tilde{x}_1$,

$$\mathcal{K}_l = 1 + \alpha_l\, q + O(q^2) \ ,$$
$$\alpha_l = N! \times \left(\tilde{A}_{d+1-l} - \sum_{l=2}^{N} \frac{N}{l} \right) \ ,$$
$$\tilde{A}_m := \sum_{1 \le m_1 < m_2 < \cdots < m_n \le N-1} \frac{N - m_1}{m_1} \cdot \frac{N - m_2}{m_2} \cdot \cdots \cdot \frac{N - m_n}{m_n} \ .$$

Appendix C

The Explicit Forms of the Kähler Potential

We calculate several explicit forms of the Kähler potential $\mathcal{K}$ in some lower dimensions for our case.

dimension	$e^{-\mathcal{K}} \cdot (\varpi_0 \bar{\varpi}_0)^{-1}$
1	$t - \bar{t}$
2	$\frac{1}{2}(t-\bar{t})^2 + \frac{1}{2}\left(\mathcal{D}_\rho t + \overline{\mathcal{D}_\rho t}\right)_{\rho=0}$
3	$\frac{1}{6}(t-\bar{t})^3 + \frac{1}{2}(t-\bar{t})\left(\mathcal{D}_\rho t + \overline{\mathcal{D}_\rho t}\right)_{\rho=0} + \frac{1}{6}\left(\mathcal{D}_\rho^2 t - \overline{\mathcal{D}_\rho^2 t}\right)_{\rho=0}$
4	$\frac{1}{24}(t-\bar{t})^4 + \frac{1}{4}(t-\bar{t})^2\left(\mathcal{D}_\rho t + \overline{\mathcal{D}_\rho t}\right)_{\rho=0} + \frac{1}{6}(t-\bar{t})\left(\mathcal{D}_\rho^2 t - \overline{\mathcal{D}_\rho^2 t}\right)_{\rho=0}$ $+\frac{1}{24}\left(\mathcal{D}_\rho^3 t + \overline{\mathcal{D}_\rho^3 t}\right)_{\rho=0} + \frac{1}{8}\left(\mathcal{D}_\rho t + \overline{\mathcal{D}_\rho t}\right)^2_{\rho=0}$

The ρ-derivatives are defined as,

$$\mathcal{D}_\rho^m t := \left(\frac{1}{2\pi i} \cdot \frac{\partial}{\partial \rho} \right)^{m+1} \cdot \log \hat{\varpi}_0(z\,;\,\rho) \ ,$$

and the "bar" means the complex conjugate of them. The above examples coincide with the well-known results of the torus ($d = 1$), the K3 surface ($d = 2$) and the Quintics ($d = 3$). In the B(W)-model case, the Kähler potential can be defined by using a holomorphic d-form Ω and an anti-holomorphic d-form $\bar{\Omega}$ as,

$$e^{-\mathcal{K}} = \int_W \Omega \wedge \bar{\Omega} \ .$$

On the other hand, this d-form can be expanded by a set of dual basis of the homology cycles $\{\gamma_l\}$,

$$\Omega = \sum_{l=0}^{d} \left(\int_{\gamma_l} \Omega \right) \gamma_l^* = \sum_{l=0}^{d} (\varpi_l) \cdot \gamma_l^* \ ,$$
$$\int_{\gamma_m} \gamma_l^* = \delta_{l,m} \quad , \quad \gamma_l \in H_d(W\,;\,\mathbf{Z}) \ .$$

We suppose that intersection numbers of these cycles are given by,

$$\mathbf{I}_{l,m} := \int_W \gamma_l^* \wedge \gamma_m^* \ ,$$

$$\mathbf{I} = \begin{pmatrix} O & & & & (-1)^d \\ & & & (-1)^{d-1} & \\ & & \cdots & & \\ & (-1)^1 & & & \\ (-1)^0 & & & & O \end{pmatrix} \ .$$

When one performs the monodromy transformation around the point $z = 0$ as $z \to \exp(2\pi\sqrt{-1}\,n) \cdot z$, the functions ϖ_l transform,

$$\varpi := {}^t\begin{pmatrix} \varpi_0 & \varpi_1 & \cdots & \varpi_d \end{pmatrix} \ ,$$

$$\varpi \to T_n \cdot \varpi \qquad (z \to \exp(2\pi\sqrt{-1}n) \cdot z) \ ,$$

$$T_n := T_1^n \ ,$$

$$T_1 := \exp \begin{pmatrix} 0 & & & & & O \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 0 & \\ O & & & & 1 & 0 \end{pmatrix} \ .$$

Then the intersection matrix $\mathbf{I}$ is affected by this matrix,

$$\mathbf{I} \to {}^tT_n \mathbf{I} T_n = \mathbf{I} \ .$$

So we presume that we have an appropriate set of homology cycles.

Using this assumption, we obtain the Kähler potential in the monodromy invariant form,

$$\begin{aligned} g_{0\bar{0}} &= e^{-\mathcal{K}} \\ &= (\varpi_0\bar{\varpi}_0) \sum_{a=0}^{d} \omega_a \bar{\omega}_{d-a} \cdot (-1)^{d-a} \\ &= (\varpi_0\bar{\varpi}_0)\ S_d\,(\, \tilde{z}_1\,,\ \tilde{z}_2\,,\ \cdots\,,\ \tilde{z}_d\,) \ , \\ \tilde{z}_m &:= \tilde{x}_m + (-1)^m \overline{\tilde{x}}_m \ , \\ \tilde{x}_n &:= \frac{1}{n!} \mathcal{D}_\rho^n \log \hat{\varpi}_0(z\,;\,\rho)\big|_{\rho=0} \ . \end{aligned}$$

Appendix D

Derivation of the Parameters α and β

In deriving the unknown parameters α and β, we need the information about the behaviour of the period ϖ_0 and the mirror map t around the points $\psi \sim 0$ or $\psi \sim \infty$. Firstly

the period ϖ_0 can be expressed as,

$$
\begin{aligned}
\varpi_0(\psi) &= \sum_{n=0}^{\infty} \frac{\Gamma(Nn+1)}{[\Gamma(n+1)]^N} \cdot \frac{1}{(N\psi)^{Nn}} \quad , \quad (|\psi|>1) \\
&= \frac{-1}{N} \sum_{r=1}^{N-1} \left(\frac{\tilde{\alpha}^r - 1}{2\pi i} \right)^{N-1} \cdot (N\psi)^r \\
&\quad \times \sum_{m=0}^{\infty} \frac{\left[\Gamma\left(m+\frac{r}{N}\right)\right]^N}{\Gamma(Nm+r)} \cdot (N\psi)^{Nm} \quad , \quad (|\psi|<1) \quad ,
\end{aligned}
$$

where $\tilde{\alpha} := \exp\left(\frac{2\pi i}{N}\right)$. It behaves as,

$$
\left\{ \begin{array}{lll} \varpi_0 & \sim & \psi^1 \qquad\qquad (\psi \sim 0) \quad , \\ \varpi_0 & \sim & \text{regular} \qquad (\psi \sim \infty) \quad . \end{array} \right. \tag{52}
$$

Secondly we can write the mirror map $t(\psi)$ as,

$$
\begin{aligned}
2\pi i \cdot t(\psi) &= N \log \frac{1}{(N\psi)} \\
&\quad + N \sum_{n=1}^{\infty} \frac{\Gamma(Nn+1)}{[\Gamma(n+1)]^N} \cdot \{\Psi(Nn+1) - \Psi(n+1)\} \cdot \frac{1}{(N\psi)^{Nn}} \quad , \quad (|\psi|>1) \\
&= -\frac{\displaystyle\sum_{r=1}^{N-1} \tilde{\alpha}^r (\tilde{\alpha}^r - 1)^{N-2} \cdot (N\psi)^r \xi_r(\psi)}{\displaystyle\sum_{s=1}^{N-1} (\tilde{\alpha}^s - 1)^{N-1} \cdot (N\psi)^s \xi_s(\psi)} \quad , \quad (|\psi|<1) \quad , \\
&\quad \xi_r(\psi) := \sum_{m=0}^{\infty} \frac{\left[\Gamma\left(m+\frac{r}{N}\right)\right]^N}{\Gamma(Nm+r)} \cdot (N\psi)^{Nm} \quad .
\end{aligned}
$$

From this formula, we can write asymptotic behaviours of t,

$$
\left\{ \begin{array}{lll} t & \sim & \text{regular} \qquad (\psi \sim 0) \quad , \\ t & \sim & \log \psi^{-N} \qquad (\psi \sim \infty) \quad . \end{array} \right.
$$

In the $\psi \sim 0$ case, the F_1 tends to a form,

$$
\mathsf{F}_1 \sim \frac{1}{2} \log \left\{ \psi^{\alpha - v} \right\} \quad .
$$

By postulating the regularity, we can obtain the α,

$$
\alpha = v \quad . \tag{53}
$$

Also the F_1 behaves as,

$$
\mathsf{F}_1 \sim \frac{1}{2} \log \left\{ \psi^{\alpha + N\beta + u} \right\} \quad ,
$$

in the $\psi \to \infty$ limit. On the other hand, by the consideration of the large radius limit, this F_1 turns to be a formula,

$$\mathsf{F}_1 \sim \frac{1}{2} \log \left\{ \psi^{N \cdot \frac{1}{12} \cdot N_{d-1}} \right\} \quad .$$

So the β can be obtained,

$$\beta = \frac{1}{12} \cdot N_{d-1} - \frac{u+v}{N} \quad . \tag{54}$$

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